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Analysis of Two Queues in Parallel with Mixed Priority Service and A Finite Population

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Abstract

The present paper deals with a generalization of the homogeneous single server finite source inventory system with two parallel finite waiting lines (waiting lines 1 and 2) and two classes of customers - one with high priority customer and the other with low priority customer. We assumed that the arriving high priority and low priority customers enter the waiting line-1 and waiting line-2 respectively. The high priority customers have a mixed priority over the low priority customers. The inventory is replenished based on (s, S) policy and the replenishing times are assumed to be exponentially distributed. The server provides two types of services - one with essential service and the other with a second optional service. The service times of the 1st (essential) and 2nd (optional) services are independent and exponentially distributed. The joint probability distribution of the number of high priority customers in the waiting line - 1, number of low priority customers in the waiting line – 2, status of the server and the inventory level is obtained in the steady state case. Some important system performance measures in the steady state are derived and the long-run total expected cost rate is also derived.

Keywords: (s, S) policy, Continuous review, Inventory with service time, Impatient customers, Priority customers, Essential and optional service, Markov process.

1. Introduction

Several researchers have studied the inventory systems in which demanded items are instantaneously distributed from stock (if available) to the customers. During stock out period, the demands of a customer are either not satisfied (lost sales case) or satisfied only after getting the receipt of the ordered items (backlog case). In the backlog case, either all demands (full backlog case) or only a limited number of demands (partial backlog case) are satisfied during stock out period. To know the review of these works see Cakanyildirim et al., [7], Dur´an et al. [8], Elango and Arivarignan[9], Goyal and Giri [10], Kalpakam and Arivarignan([14], [15]), Liu and Yang[17], Nahmias[18], Raafat[20] and Yadavalli et al.[22] and the references therein.

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However, in the case of inventories maintained at service facilities, after some service is performed on the demanded items they are distributed to the customers. In such situations, the items are issued not on demanding rather it is done after a random time of service. It causes the formation of queues in front of service centres. As a result there is a need for study of both the inventory level and the queue length in the long run. Berman and Kim [2] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [5] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [6] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [3] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [4] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. Krishnamoorthy et al., [16] introduced an additional control policy (N-policy) into (s, S) inventory system with positive service time.

In all the above models, the authors assumed that after completion of the service (namely, regular service or main service or essential service), immediately the customers leave the system. But in many real life situations, all the arriving customers first require an essential service and some of them may require the secondary service provided by the same server. In queueing systems the latter type of service referred to as second optional service. The concept of the second optional service with queue has been studied by several researchers in the past. As a related work we refer [12, 13].

An important issue in the queueing-inventory system with two types of customers is the priority assignment problem. For example, in assembly manufacturing system customers with long-term supply contracts have been given high priority than the other ordinary customers. In multi-specialty hospitals patients with serious illness are given high priority than the other patients opting for routine check or else. These real life problems stimulate us to study the queueing-inventory systems with two types of customers. The high priority customers have either preemptive or non-preemptive priority over the low priority customers. Artalejo et al. [1] analyzed queue with retrial customers looked to have preemptive priority over the waiting room.

Ning Zhao and Zhaotong Lian [19] analyzed a queueing-inventory system with two classes of customers. The authors have assumed the arrival of the two-types of customer's form independent Poisson processes and exponential service times. Each service uses one item from the attached inventory supplied by an outside supplier with exponentially distributed lead time. Jeganathan et al. [11] studied a retrial inventory system with non-preemptive priority service. The authors have assumed the arrival of the two-types of customers form independent with Poisson arrival and exponential service times. Retrial is introduced for low priority customers only.

The joint probability distribution of the number of high priority customers in the waiting area, the number of low priority customers in the orbit and the inventory level is obtained for the steady state case. It may be noted that recently yadavalli et.al [23] analyzed a retrial inventory system with impatient customers. Also, they assumed the low priority customers request overhaul of the item only and the high priority customers demand unit item, which is delivered after performing service on the item.

In this paper, we introduce the mixed priority concept in the context of a queueing - inventory models with two parallel finite waiting lines, second optional service and a finite population. Our research work on finite source inventory system is motivated by the service facility system with restricted customers for example military canteen providing service to soldiers or a company canteen serving the members of the specific working community in the company. Machine service problems within an industry are also a problem which motivated us to create the present stochastic model. The problem we consider is more relevant to the real life situation. The joint probability distribution of the number of high priority customers in the waiting line 1, the number of low priority customers in the waiting line 2 and the inventory level is obtained for the steady state case. Various measures of system performance are computed in the steady state case.

The rest of the paper is organized as follows. In the next section, the problem formulation and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are obtained in section 3. Some key system performance measures are derived in section 4. In section 5, we have derived the total expected cost rate. The last section is meant for conclusion.

2. Mathematical Modelling

In this paper, we considered the stochastic model of inventory system with service facility and the demands originated from a finite population of sources K, $0 < K < \infty$. Maximum inventory level is denoted by S and the inventory is replenished according to (s, S) ordering policy. According to this policy, the reorder level is fixed as s and an order is placed when the inventory level reaches the reorder level. The ordering quantity is $Q(= S - s > s + 1)$ items. The condition $S - s > s + 1$ ensures that no perpetual shortage in the stock after replenishment. The lead time is assumed to be exponential with parameter $\beta(> 0)$. Customers arriving at the service station belong to any one of the two types such that the high priority and the low priority customers and their arrivals belong to independent quasi-random distributions with parameters λ_1 and λ_2 respectively.

The service facility consists of a single server and two finite waiting lines (waiting lines 1 and 2). The waiting line 1 is designed for high priority customers and its maximum size is

N. The waiting line 2 is designed for low priority customers and its maximum size is *M*=*K*-*N*. Each of the customers is, at any time, either in the source or in his respective waiting line. The items are issued to the demanding customers only after some random time due to some service on it. The service time of the customer depends on the priority level of the customer.

The 'Mixed Priority'(i.e., preemptive priority and non-preemptive priority)' concept plays an important role here. That is, the arriving low priority customer finds both the two waiting lines are empty and the server is idle with the positive inventory level is immediately taken to service, otherwise he should wait for getting service in his waiting line. During the time of low priority customer's service, with probability q, any arriving high priority customer interrupts the low priority customer's service also he forced him to wait in his waiting line and starts his service. It may be called as preemptive priority service. After the completion of the essential service of the high priority customer, he requires for second optional service with probability p or he leaves the system with probability $1-p$. The additional optional service is offered only for high priority customers. The interrupted low priority customer's service will resume only after the completion of the high priority customer's service. During the time of low priority customer's service, with probability $1-q$, any arriving high priority customer decides to wait in the waiting line 1 and to get service after the completion of the low priority customer's service. This type of service may be called as non-preemptive priority service. The head of the customer in the waiting line 1 is immediately taken to service after the present low/high priority customer's service, irrespective of the number of customers in the waiting line 1. The service times of the high priority customer, low priority customer and the optional service are independent and assumed to follow an exponential distribution with parameter μ_1, μ_2 and μ_3 respectively. During the service time of the high priority customer, any arriving low priority customer joins the waiting line 1. Any arriving high priority customer or low priority customer, who finds his waiting line full are considered to be lost.

Notation:

e : acolumnvectorofappropriatedimensioncontainingallones

0 : Zeromatrix

$[A]_{ij}$: entryat $(i,j)^{th}$ positionofamatrixA

$$\delta_{ij} : \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

$\bar{\delta}_{ij} : 1-\delta_{ij}$

$k \in V_i^j : k = i, i+1, \ldots j$

3. Markov chain

Let $L(t)$, $X_1(t)$ and $X_2(t)$ respectively, denote the inventory level, the number of high priority customers in the waiting line 1 and the number of low priority customers in the waiting line 2 at time *t*.

Further, the server status Y (t) is defined as follows:

$$Y(t): \begin{cases} 3, & \text{if the server is providing essential service to a low priority customer at time t} \\ 2, & \text{if the server is providing second optional service to a high priority customer at time t} \\ 1, & \text{if the server is providing essential service to a high priority customer at time t} \\ 0, & \text{if the server is idle at time t} \end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $I(t) = \{(L(t), Y(t), X_1(t), X_2(t)), t \geq 0\}$ is a continuous time Markov chain with state space given by $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, where

$$E_1 : \{(i_1, 0, 0, 0) \mid i_1 = 1, 2, \ldots, S,\}$$

$$E_2 : \{(0, 0, i_3, i_4) \mid i_3 = 0, 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M,\}$$

$$E_3 : \{(i_1, 1, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, i_3 = 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M,\}$$

$$E_4 : \{(i_1, 2, i_3, i_4) \mid i_1 = 0, 1, 2, \ldots, S, i_3 = 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M,\}$$

$$E_5 : \{(i_1, 3, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, i_3 = 0, 1, 2, \ldots, N, i_4 = 1, 2, \ldots, M,\}$$

By ordering the state space ($\ll 0 \gg, \ll 1 \gg, \ldots, \ll S \gg$), the infinitesimal generator Θ can be conveniently written in a block partitioned matrix with entries

$$[\Theta] = \begin{cases} A_{i_1 j_1}, & j_1 = i_1, \quad i_1 \in V_0^S, \\ 0, & otherwise. \end{cases}$$

More explicitly,

Due to the assumptions made on the demand and replenishment processes, we note that

$$\Theta_{i_1, j_1} = 0, \quad for \quad j_1 \neq i_1, i_1 - 1, i_1 + Q.$$

We first consider the case Θ_{i_1, i_1+Q}. This will occur only when the inventory level is replenished. First we consider the inventory level is zero, that is $\Theta_{0,Q}$. For this

Case (i) When there is no customer in both the waiting hall 1 and 2, at the time of replenishment the state of the system changes from (0,0,0,0) to (*Q*,0,0,0), with intensity of transition b. The sub matrix of the transition rates from <<0, 0>> to <<Q, 0>>, is given by

$$[C_{00}^{(1)}]_{i_3 j_3} = \begin{cases} C_6, & j_3 = 0, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[C_6]_{i_4 j_4} = \begin{cases} \beta, & j_4 = 0, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

Case (ii) When there is a no customer in the waiting hall 1 and the waiting hall 2 has at least one customer, at the time of replenishment takes the system state from $(0, 0, 0, i_4)$ to $(Q, 3, 0, i_4)$, $i_4 = 1, 2, \ldots, M$. The sub matrix of the transition rates from <<0,0>> to << Q , 3>>, is given by $C_{03}^{(1)}$

$$[C_{03}^{(1)}]_{i_3 j_3} = \begin{cases} C_3, & j_3 = 0, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[C_3]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

Case (iii) When each waiting hall has at least one customer, the replenishment takes the system state from $(0, 0, i_3, i_4)$ to $(Q, 1, i_3, i_4)$ or from $(0, 2, i_3, i_4)$ to $(Q, 2, i_3, i_4)$, $i_3 = 1, 2, \ldots, N$, $i_4 = 1, 2, \ldots, M$. The sub matrix of the transition rates from <<0,0>> to << Q, 1>> and <<0,2>> to <<Q,2>> is given by $C_{01}^{(1)}$ and $C_{22}^{(1)}$ respectively.

$$[C_{01}^{(1)}]_{i_3 j_3} = \begin{cases} C_2, & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & otherwise, \end{cases}$$

$$[C_{22}^{(1)}]_{i_3 j_3} = \begin{cases} C_2, & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & otherwise, \end{cases}$$

Hence

$$[\Theta_{0,Q}]_{i_2 j_2} = \begin{cases} C_{00}^{(1)}, & j_2 = 0, \quad i_2 = 0, \\ C_{01}^{(1)}, & j_2 = 1, \quad i_2 = 0, \\ C_{03}^{(1)}, & j_2 = 3, \quad i_2 = 0, \\ C_{22}^{(1)}, & j_2 = 2, \quad i_2 = 2, \\ 0, & otherwise, \end{cases}$$

We denote $\Theta_{0,Q}$ as C_1.

We now consider the case when the inventory level lies between one to s. We note that for this case, only the inventory level changes from i_1 to i_1+Q. The other system states does not change. Hence

$$[\Theta_{i_1,i_1+Q}]_{i_2 j_2} = \begin{cases} C_{00}^{(0)}, & j_2 = 0, \quad i_2 = 0, \\ C_{11}^{(0)}, & j_2 = 1, \quad i_2 = 1, \\ C_{22}^{(0)}, & j_2 = 2, \quad i_2 = 2, \\ C_{33}^{(0)}, & j_2 = 3, \quad i_2 = 3, \\ 0, & otherwise, \end{cases}$$

where, A_{i_1,i_1+Q} is denoted by C.

Next, we consider the case $\Theta_{i_1,i_1-1}, i_1 = 1,2,\ldots,S.$ This will occur only when either the essential service completion of a high priority customer or a low priority customer. For this, we have the following cases occur:

Case (i): If the server is providing essential service to a high priority customer and only one item in the inventory

- at the time of the essential service completion of a high priority customer both the inventory level and waiting hall 1 size decrease by one and the server become idle. The intensity of this transition is given by $(1-p)\mu_1$. Otherwise, if the essential service of a customer is completed then he may ask second optional service, in which case his additional service will immediately commence. The intensity of this transition is given by $p\mu_1$.

Case (ii): If the server is providing service to a low priority customer and only one item in the inventory

- at the time of the service completion of a low priority customer both the inventory level and waiting hall 2 size decrease by one and the server become idle. The intensity of this transition is given by μ_3.

Hence $\Theta_{1,0}$ is given by

$$[\Theta_{1,0}]_{i_2 j_2} = \begin{cases} F_{10}^{(3)}, & j_2 = 0, \quad i_2 = 1, \\ F_{12}^{(3)}, & j_2 = 2, \quad i_2 = 1, \\ F_{30}^{(3)}, & j_2 = 0, \quad i_2 = 3, \\ 0, & otherwise, \end{cases}$$

$$[F_{10}^{(3)}]_{i_3 j_3} = \begin{cases} D_0, & j_3 = i_3 - 1, \quad i_3 \in V_1^N, \\ 0, & otherwise, \end{cases}$$

$$[F_{12}^{(3)}]_{i_3 j_3} = \begin{cases} D_1, & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & otherwise, \end{cases}$$

$$[F_{30}^{(3)}]_{i_3 j_3} = \begin{cases} W, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & otherwise, \end{cases}$$

$$[W]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = i_4 - 1, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$D_0 = (1-p)\mu_1 I_{(M+1)}, D_1 = p\mu_1 I_{(M+1)}$$

We $\Theta_{1,0}$ is denoted by F_3.

Case (iii): If the server is providing essential service to a high priority customer, the inventory level is more than one, only one customer in the waiting line 1 and no customer in the waiting line 2

- the essential service of a high customer is completed then with probability *p* the customer may ask second optional service, in which case his additional service will immediately commence and the state of the process from $(i_1,1,1,0)$ to $(i_1-1,2,1,0)$. Otherwise, with probability (1 −p) he may opt to leave the system, in which case the server become idle and the state of the system changes from $(i_1,1,1,0)$ to $(i_1-1,0,0,0)$.

Case (iv): If the server is providing essential service to a high priority customer, the inventory level is more than one, only one customer in the waiting line 1 and at least one customer in the waiting line 2

- the essential service of a high customer is completed then the customer may leave the system without asking additional service, in which case the low priority customer at the head of the waiting line 2 is taken up for his service and the state of the process from $(i_1,1,1,i_4)$ to $(i_1-1,3,0,i_4)$.

Case (v): If the server is providing essential service to a high priority customer, the inventory level is more than one, waiting line 1 has more than one customer and the number of customers in the waiting line 2 lies between zero to M,

- the essential service of a high customer is completed then with probability (1 − p) he may opt to leave the system, in which case the another high priority customer at the head of the waiting line 1 is taken up for his first essential service and the state of the process from $(i_1,1,i_3,i_4)$ to $(i_1-1,1,i_3-1,i_4)$ Otherwise, with probability p the customer may ask second optional service and the state of the process from $(i_1,1,i_3,i_4)$ to $(i_1-1,2,i_3,i_4)$.

Case (vi): If the server is busy with a low priority customer, the inventory level is more than one, no customer in the waiting line 1 and only one customer in the waiting line 2,

- after the service completion of a low priority customer, the server become idle and he is ready to serve any arriving customers. The state of the system changes from $(i_1,3,0,1)$ to $(i_1-1,0,0,0)$.

Case (vii): If the server is busy with a low priority customer, the inventory level is more than one, no customer in the waiting line 1 and more than one customer in the waiting line 2,

- after the service completion of a low priority customer, the server take another low priority customer for his service at the head of the waiting line 2 and the state of the process from $(i_1,3,0,i_4)$ to $(i_1-1,3,0,i_4-1)$.

Case (viii): If the server is busy with a low priority customer, the inventory level is more than one and each waiting line has at least one customer,

- after the service completion of a low priority customer, the server will immediately become busy with a high priority customer and the state of the process from $(i_1, 3, i_3, i_4)$ to $(i_1 - 1, 1, i_3, i_4 - 1)$.

Hence $\Theta_{i_1, i_1 - 1}, i_1 = 2, 3, \ldots, S,$ is given by

$$[\Theta_{i_1, i_1-1}]_{i_2 j_2} = \begin{cases} F_{10}^{(4)}, & j_2 = 0, \quad i_2 = 1, \\ F_{11}^{(4)}, & j_2 = 1, \quad i_2 = 1, \\ F_{12}^{(4)}, & j_2 = 2, \quad i_2 = 1, \\ F_{13}^{(4)}, & j_2 = 3, \quad i_2 = 1, \\ F_{30}^{(4)}, & j_2 = 0, \quad i_2 = 3, \\ F_{31}^{(4)}, & j_2 = 1, \quad i_2 = 3, \\ F_{33}^{(4)}, & j_2 = 3, \quad i_2 = 3, \\ 0, & otherwise, \end{cases}$$

$$[F_{10}^{(4)}]_{i_3 j_3} = \begin{cases} W_0, & j_3 = 0, \quad i_3 = 1, \\ 0, & otherwise, \end{cases}$$

$$[W_0]_{i_4 j_4} = \begin{cases} (1-p)\mu_1, & j_4 = 0, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{11}^{(4)}]_{i_3 j_3} = \begin{cases} D_0, & j_3 = i_3 - 1, \quad i_3 \in V_2^N, \\ 0, & otherwise, \end{cases}$$

$$[F_{12}^{(4)}]_{i_3 j_3} = \begin{cases} D_1, & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & otherwise, \end{cases}$$

$$[F_{13}^{(4)}]_{i_3 j_3} = \begin{cases} W_2, & j_3 = 0, \quad i_3 = 1, \\ 0, & otherwise, \end{cases}$$

$$[W_2]_{i_4 j_4} = \begin{cases} (1-p)\mu_1, & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$[F_{30}^{(4)}]_{i_3 j_3} = \begin{cases} W_1, & j_3 = 0, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[W_1]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = 0, \quad i_4 = 1, \\ 0, & otherwise, \end{cases}$$

$$[F_{31}^{(4)}]_{i_3 j_3} = \begin{cases} W, & j_3 = i_3 - 1, \quad i_3 \in V_1^N, \\ 0, & otherwise, \end{cases}$$

$$[F_{33}^{(4)}]_{i_3 j_3} = \begin{cases} W_3, & j_3 = 0, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[W_3]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = i_4 - 1, \quad i_4 \in V_2^M, \\ 0, & otherwise, \end{cases}$$

We will denote $\Theta_{i_1, i_1 - 1}, i_1 = 2,3,\ldots,S,$ as F_4.

Finally, we consider the case $\Theta_{i_1, i_1}, i_1 = 0,1,\ldots,S.$ Here due to each one of the following mutually exclusive cases, a transition results:

- an arrival of a high priority customer may occur
- an arrival of a low priority customer may occur
- a second optional service of high customer may be completed

When the inventory level is zero and server is idle, we have the following state changes may arise:

Case (a):

- an arrival of a high priority customer increases the number of customer waiting in the waiting hall 1 increases by one and the state of the arrival process moves $(0,0,i_3,i_4)$ to $(0,0,i_3+1,i_4)$, $i_3=0,1,\ldots,N-1$; $i_4=0,1,\ldots,M$ with intensity of transition $(K-(i_3+i_4))\lambda_1$.
- a low priority customer enters the waiting hall 2 which takes the state of the system from $(0,0,i_3,i_4)$ to $(0,0,i_3,i_4+1)$, $i_3=0,1,\ldots,N$; $i_4=0,1,\ldots,M-1$, with intensity of transition $(K-(i_3+i_4))\lambda_2$.

When the inventory level is zero and the server is providing second optional service to a high priority customer, we have the following state changes may arise:

Case(b):

- at the time of second optional service completion of a high priority customer, the customer level in the waiting hall 1 decrease by one and the server become idle. The state of the system moves from $(0,2,i_3,i_4)$ to $(0,0,i_3-1,i_4)$, $i_3=1,\ldots,N; i_4=0,1,\ldots,M$, with intensity of transition μ_2. The sub matrix of this transition rate is given by <<0, 2>> to <<0, 0>> is $\mu_2 I_{(M+1)}$ and is denoted by A_0.

- the remaining arguments are same as the above case (a).

The transition rates for any other transitions not considered above, when the inventory level is zero, are zero. The intensity of passage in the state $(0,i_2,i_3,i_4)$ is given by

$$-\sum_{(0,i_2,i_3,i_4)\neq(0,j_2,j_3,j_4)} a((i_1,i_2,i_3,i_4),(j_1,j_2,j_3,j_4))$$

Using the above arguments, we have constructed the following matrices

$$[B]_{i_4 j_4}=\begin{cases}(K-(i_3+i_4))\lambda_2, & j_4=i_4+1, \quad i_4\in V_0^{M-1},\\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\overline{\delta}_{i_4 M})+\beta), & j_4=i_4, \quad i_4\in V_0^{M},\\ 0, & otherwise,\end{cases}$$

$$[B_0]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\overline{\delta}_{i_4 M} + \beta), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[B_1]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\overline{\delta}_{i_4 M}) + \beta + \mu_3), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[B_2]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\overline{\delta}_{i_4 M} + \mu_3 + \beta), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

Combining theses matrices in suitable form, we get

$$[F_{00}^{(0)}]_{i_3 j_3} = \begin{cases} A, & j_3 = i_3+1, \quad i_3 \in V_0^{N-1}, \\ B, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ B_0, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[F_{22}^{(0)}]_{i_3 j_3} = \begin{cases} A, & j_3 = i_3+1, \quad i_3 \in V_1^{N-1}, \\ B_1, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ B_2, & j_3 = N, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[F_{20}^{(0)}]_{i_3 j_3} = \begin{cases} A_0, & j_3 = i_3-1, \quad i_3 \in V_1^{N}, \\ 0, & otherwise, \end{cases}$$

Hence the matrix A_{00} is given by

$$[A_{00}]_{i_2 j_2} = \begin{cases} F_{00}^{(0)}, & j_2 = 0, \quad i_2 = 0, \\ F_{22}^{(0)}, & j_2 = 2, \quad i_2 = 2, \\ F_{20}^{(0)}, & j_2 = 0, \quad i_2 = 2, \\ 0, & otherwise, \end{cases}$$

and is denoted by F_0. Arguments similar to above yields.

For $i_1 = 1, 2, \ldots, s$,

$$[A_{i_1 i_1}]_{i_2 j_2} = \begin{cases} F_{00}^{(1)}, & j_2 = 0, \quad i_2 = 0, \\ F_{01}^{(1)}, & j_2 = 1, \quad i_2 = 0, \\ F_{11}^{(1)}, & j_2 = 1, \quad i_2 = 1, \\ F_{03}^{(1)}, & j_2 = 3, \quad i_2 = 0, \\ F_{20}^{(1)}, & j_2 = 0, \quad i_2 = 2, \\ F_{21}^{(1)}, & j_2 = 1, \quad i_2 = 2, \\ F_{22}^{(1)}, & j_2 = 2, \quad i_2 = 2, \\ F_{23}^{(1)}, & j_2 = 3, \quad i_2 = 2, \\ F_{31}^{(1)}, & j_2 = 1, \quad i_2 = 3, \\ F_{33}^{(1)}, & j_2 = 3, \quad i_2 = 3, \\ 0, & otherwise, \end{cases}$$

$$[F_{00}^{(1)}]_{i_3 j_3} = \begin{cases} F_5, & j_3 = 0, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_5]_{i_4 j_4} = \begin{cases} -((K - (i_3 + i_4))(\lambda_1 + \lambda_2) + \beta), & j_4 = 0, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{01}^{(1)}]_{i_3 j_3} = \begin{cases} A_1, & j_3 = 1, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[A_1]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_1, & j_4 = 0, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{11}^{(1)}]_{i_3 j_3} = \begin{cases} A, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ G_3, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ G_4, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[G_3]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1 + \lambda_2 \overline{\delta}_{i_4 M}) + \mu_1 + \beta), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$G_4]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2 \overline{\delta}_{i_4 M} + \mu_1 + \beta), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[F_{03}^{(1)}]_{i_3 j_3} = \begin{cases} A_2, & j_3 = 0, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[A_2]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2 & j_4 = 0, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{20}^{(1)}]_{i_3 j_3} = \begin{cases} A_3, & j_3 = 0, \quad i_3 = 1, \\ 0, & otherwise, \end{cases}$$

$$[A_3]_{i_4 j_4} = \begin{cases} \mu_3 & j_4 = 0, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{21}^{(1)}]_{i_3 j_3} = \begin{cases} A_0, & j_3 = i_3 - 1, \quad i_3 \in V_2^N, \\ 0, & otherwise, \end{cases}$$

$$[F_{22}^{(1)}]_{i_3 j_3} = \begin{cases} A, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ G_5, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ G_6, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[G_5]_{i_4 j_4} = \begin{cases} (K - (i_3 + i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K - (i_3 + i_4))(\lambda_1 + \lambda_2 \overline{\delta}_{i_4 M}) + \mu_3 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & otherwise, \end{cases}$$

$$[G_6]_{i_4 j_4} = \begin{cases} (K - (i_3 + i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K - (i_3 + i_4))\lambda_2 \overline{\delta}_{i_4 M} + \mu_3 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & otherwise, \end{cases}$$

$$[F_{23}^{(1)}]_{i_3 j_3} = \begin{cases} J, & j_3 = 0, \quad i_3 = 1, \\ 0, & otherwise, \end{cases}$$

$$[J]_{i_4 j_4} = \begin{cases} \mu_3, & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$[F_{31}^{(1)}]_{i_3 j_3} = \begin{cases} G_0, & j_3 = 1, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[G_0]_{i_4 j_4} = \begin{cases} q(K-(i_3+i_4))\lambda_1 & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$[F_{33}^{(1)}]_{i_3 j_3} = \begin{cases} G_1, & j_3 = 1, \quad i_3 = 0, \\ G_2, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ H, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ H_1, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[G_1]_{i_4 j_4} = \begin{cases} (1-q)(K-(i_3+i_4))\lambda_1 & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$[G_2]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_1, & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$[H]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_1^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1 + \lambda_2 \overline{\delta}_{i_4 M}) + \mu_2 + \beta), & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

$$[H_1]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_1^{M-1}, \\ -((K-(i_3+i_4))\lambda_2 \overline{\delta}_{i_4 M} + \mu_2 + \beta), & j_4 = i_4, \quad i_4 \in V_1^M, \\ 0, & otherwise, \end{cases}$$

For $i_1 = s+1, s+2, \ldots, S,$

$$[F_2]_{i_2 j_2} = \begin{cases} F_{00}^{(2)}, & j_2 = i_2, \quad i_2 = 0, \\ F_{01}^{(2)}, & j_2 = 1, \quad i_2 = 0, \\ F_{03}^{(2)}, & j_2 = 3, \quad i_2 = 0, \\ F_{11}^{(2)}, & j_2 = i_2, \quad i_2 = 1, \\ F_{20}^{(2)}, & j_2 = 0, \quad i_2 = 2, \\ F_{21}^{(2)}, & j_2 = i_2, \quad i_2 = 2, \\ F_{22}^{(2)}, & j_2 = 2, \quad i_2 = 2, \\ F_{23}^{(2)}, & j_2 = 3, \quad i_2 = 2, \\ F_{31}^{(2)}, & j_2 = 3, \quad i_2 = 3, \\ F_{33}^{(2)}, & j_2 = 3, \quad i_2 = 3, \\ 0, & otherwise, \end{cases}$$

$$[F_{00}^{(2)}]_{i_3 j_3} = \begin{cases} F_6, & j_3 = i_3, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_6]_{i_4 j_4} = \begin{cases} -K(\lambda_1 + \lambda_2), & j_4 = i_4, \quad i_4 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{01}^{(2)}]_{i_3 j_3} = \begin{cases} A_1, & j_3 = 1, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{11}^{(2)}]_{i_3 j_3} = \begin{cases} A, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ H_2, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ H_3, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[H_2]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\overline{\delta}_{i_4 M})+\mu_1), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[H_3]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\overline{\delta}_{i_4 M}+\mu_1), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[F_{20}^{(2)}]_{i_3 j_3} = \begin{cases} A_3, & j_3 = 0, \quad i_3 = 1, \\ 0, & otherwise, \end{cases}$$

$$[F_{21}^{(2)}]_{i_3 j_3} = \begin{cases} A_0, & j_3 = i_3 - 1, \quad i_3 \in V_2^{N}, \\ 0, & otherwise, \end{cases}$$

$$[F_{22}^{(2)}]_{i_3 j_3} = \begin{cases} A, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ H_4, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ H_5, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[F_{23}^{(2)}]_{i_3 j_3} = \begin{cases} J, & j_3 = 0, \quad i_3 = 1, \\ 0, & otherwise, \end{cases}$$

$$[H_4]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\overline{\delta}_{i_4 M})+\mu_3), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[H_5]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\overline{\delta}_{i_4 M} + \mu_3), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & otherwise, \end{cases}$$

$$[F_{31}^{(2)}]_{i_3 j_3} = \begin{cases} G_0, & j_3 = 1, \quad i_3 = 0, \\ 0, & otherwise, \end{cases}$$

$$[F_{33}^{(2)}]_{i_3 j_3} = \begin{cases} G_1, & j_3 = 1, \quad i_3 = 0, \\ G_2, & j_3 = i_3+1, \quad i_3 \in V_1^{N-1}, \\ H_6, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ H_7, & j_3 = i_3, \quad i_3 = N, \\ 0, & otherwise, \end{cases}$$

$$[H_6]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_1^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\overline{\delta}_{i_4 M}) + \mu_2), & j_4 = i_4, \quad i_4 \in V_1^{M}, \\ 0, & otherwise, \end{cases}$$

$$[H_7]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_1^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\overline{\delta}_{i_4 M} + \mu_2), & j_4 = i_4, \quad i_4 \in V_1^{M}, \\ 0, & otherwise, \end{cases}$$

We denote $A_{i_1 i_1}, i_1 = 1,2,\ldots,s$ as F_1 and $A_{i_1 i_1}, i_1 = s+1, s+2,\ldots,S$ as F_2. Hence the matrix Θ can be written in the following form

$$[\Theta]_{i_1 j_1} = \begin{cases} F_3, & j_1 = i_1 - 1, \quad i_1 = 1, \\ F_4, & j_1 = i_1 - 1, \quad i_1 \in V_2^S, \\ C, & j_1 = i_1 + Q, \quad i_1 \in V_1^s, \\ C_1, & j_1 = i_1 + Q, \quad i_1 = 0, \\ F_0, & j_1 = i_1, \quad i_1 = 0, \\ F_1, & j_1 = i_1, \quad i_1 \in V_1^s, \\ F_2, & j_1 = i_1, \quad i_1 \in V_{s+1}^S, \\ 0, & otherwise. \end{cases}$$

It can be noted that the matrices F_1, F_2, F_4 and C are square matrices of size $(3N+1)M+(2N+1)$. F_3 and C_1 are matrices of size $(3N+1)M+(2N+1)\times(2N+1)(M+1)$ and $(2N+1)(M+1)\times(3N+1)M+(2N+1)$ respectively. F_0 and $F_{00}^{(0)}$ are square matrices of size $(2N+1)(M+1)$ and $(N+1)(M+1)$ respectively. $F_{12}^{(3)}$, $F_{12}^{(4)}$, $F_{11}^{(4)}$, $F_{22}^{(0)}$, $F_{11}^{(1)}$, $F_{21}^{(1)}$, $F_{22}^{(1)}$, $F_{11}^{(2)}$, $F_{21}^{(2)}$, $F_{22}^{(2)}$, $C_{11}^{(0)}$, $C_{22}^{(0)}$ and $C_{22}^{(1)}$ are square matrices of size $N(M+1)$. D_0, D_1, C_2, A, B, B_0, A_0, B_1, B_2, G_3, G_4, G_5, G_6, H_2, H_3, H_4 and H_5 are square matrices of size $(M+1)$. C_4, W_3, G_1, G_2, H, H_1, H_6 and H_7 are square matrices of size M. C_5, F_4, F_5, $C_{00}^{(0)}$, $F_{00}^{(1)}$ and $F_{00}^{(2)}$ are square matrices of size 1. $C_{33}^{(0)}$, $F_{33}^{(1)}$, $F_{33}^{(2)}$ and $F_{33}^{(4)}$ are square matrices of size $(N+1)M$. $F_{13}^{(4)}$, $F_{23}^{(1)}$ and $F_{23}^{(2)}$ are matrices of size $N(M+1)\times(N+1)M$. $F_{31}^{(4)}$, $F_{31}^{(1)}$ and $F_{31}^{(2)}$ are matrices of size $(N+1)M\times N(M+1)$. $F_{10}^{(4)}$, $F_{20}^{(1)}$ and $F_{20}^{(2)}$ are matrices of size $N(M+1)\times 1$. J, C_3 and W_2 are matrices of size $(M+1)\times M$. W and G_0 are matrices of size $M\times(M+1)$. W_0, C_6 and A_3 are matrices of size $(M+1)\times 1$. $F_{01}^{(1)}$ and $F_{01}^{(2)}$ are matrices of size $1\times N(M+1)$. $F_{03}^{(1)}$ and $F_{03}^{(2)}$ are matrices of size $1\times(N+1)M$. A_1, A_2, W_1, $F_{30}^{(4)}$, $C_{00}^{(1)}$, $C_{01}^{(1)}$, $C_{03}^{(1)}$ and $F_{10}^{(3)}$ are matrices of size $1\times(M+1)$, $1\times M$, $M\times 1$, $(N+1)M\times 1$, $(N+1)(M+1)\times 1$, $(N+1)(M+1)\times N(M+1)$, $(N+1)(M+1)\times(N+1)M$ and $N(M+1)\times(N+1)(M+1)$ respectively.

3.1 Steady State Analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), Y(t), X_1(t), X_2(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1,i_2,i_3,i_4)} = \lim_{t\to\infty} Pr[L(t)=i_1, Y(t)=i_2, X_1(t)=i_3, X_2(t)=i_4 \mid L(0),Y(0),X_1(0),X_2(0)].$$

exists. Let $\mathbf{\Phi} = (\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(S)})$, each vector $\phi^{(i_1)}$ being partitioned as follows:

$$\phi^{(0)} = (\phi^{(0,0)}, \phi^{(0,2)}),$$

$$\phi^{(i_1)} = (\phi^{(i_1,0)}, \phi^{(i_1,1)}, \phi^{(i_1,2)}, \phi^{(i_1,3)}), \quad i_1 = 1,2,3,\ldots,S;$$

where

$$\phi^{(0,0)} = (\phi^{(0,0,0)}, \phi^{(0,0,1)}, \ldots, \phi^{(0,0,N)}),$$

$$\phi^{(0,2)} = (\phi^{(0,2,1)}, \phi^{(0,2,2)}, \ldots, \phi^{(0,2,N)}),$$

$$\phi^{(i_1,0)} = (\phi^{(i_1,0,0)}),$$

$$\phi^{(i_1,1)} = (\phi^{(i_1,1,1)}, \phi^{(i_1,1,2)}, \ldots, \phi^{(i_1,1,N)}), i_1 = 1,2,3,\ldots,S;$$

$$\phi^{(i_1,2)} = (\phi^{(i_1,2,1)}, \phi^{(i_1,2,2)}, \ldots, \phi^{(i_1,2,N)}), i_1 = 1,2,3,\ldots,S;$$

$$\phi^{(i_1,3)} = (\phi^{(i_1,3,0)}, \phi^{(i_1,3,1)}, \ldots, \phi^{(i_1,3,N)}), i_1 = 1,2,3,\ldots,S;$$

Further the above vectors also partitioned as follows:

$$\phi^{(0,0,i_3)} = (\phi^{(0,0,i_3,0)}, \phi^{(0,0,i_3,1)}, \ldots, \phi^{(0,0,i_3,M)}), \quad i_3 = 0,1,2,3,\ldots,N;$$

$$\phi^{(0,2,i_3)} = (\phi^{(0,2,i_3,0)}, \phi^{(0,2,i_3,1)}, \ldots, \phi^{(0,2,i_3,M)}), \quad i_3 = 1,2,3,\ldots,N;$$

$$\phi^{(i_1,0,0)} = (\phi^{(i_1,0,0,0)}), \qquad i_1 = 1,2,3,\ldots,S;$$

$$\phi^{(i_1,1,i_3)} = (\phi^{(i_1,1,i_3,0)}, \phi^{(i_1,1,i_3,1)}, \ldots, \phi^{(i_1,1,i_3,M)}), \ i_1 = 1,2,3,\ldots,S; i_3 = 1,2,3,\ldots,N;$$

$$\phi^{(i_1,2,i_3)} = (\phi^{(i_1,2,i_3,0)}, \phi^{(i_1,2,i_3,1)}, \ldots, \phi^{(i_1,2,i_3,M)}), \ i_1 = 1,2,3,\ldots,S; i_3 = 1,2,3,\ldots,N;$$

$$\phi^{(i_1,3,i_3)} = (\phi^{(i_1,3,i_3,1)}, \phi^{(i_1,3,i_3,2)}, \ldots, \phi^{(i_1,3,i_3,M)}), \ i_1 = 1,2,3,\ldots,S; i_3 = 0,1,2,3,\ldots,N;$$

Then the steady state probability $\mathbf{\Phi}$ satisfies

$$\Phi\Theta = 0 \quad \textit{and}$$

$$\sum\sum\sum_{(i_1,i_2,i_3,i_4)} \phi^{(i_1,i_2,i_3,i_4)} = 1.$$

The equation (1) yields the following set of equations:

$$\phi^{i_1} F_0 + \phi^{i_1+1} F_3 = 0, \qquad i_1 = 0,$$

$$\phi^{i_1} F_1 + \phi^{i_1+1} F_4 = 0, \qquad i_1 = 1,2,\ldots,s,$$

$$\phi^{i_1} F_2 + \phi^{i_1+1} F_4 + \phi^{(i_1-1-Q)} C_1 = 0, \qquad i_1 = s+1,\ldots,Q-1,$$

$$\phi^{0} C_1 + \phi^{i_1} F_2 + \phi^{(i_1+1)} F_4 = 0, \qquad i_1 = Q \quad (*)$$

$$\phi^{i_1-Q} C + \phi^{i_1} F_2 + \phi^{(i_1+1)} F_4 = 0, \qquad i_1 = Q+1,\ldots,S-1$$

$$\phi^{i_1-Q} C + \phi^{i_1} F_2 = 0, \qquad i_1 = S.$$

The steady state probability distribution $\phi^{i_1}, i_1 = 0,1,2,\ldots,S,$ can be obtained using the following algorithm.

Algorithm:

Solve the following system of equations to find the value of ϕ^{Q}

$$\phi^{(Q)}\Big[(-1)^{Q}(F_4F_2^{-1})^{(Q-(s+1))}(F_4F_1^{-1})^{s}(F_3F_0^{-1})C_1 + F_2 + (-1)^{Q}\sum_{j=0}^{s-1}(F_4F_2^{-1})^{(2(s-1)-j)}(F_4F_1^{-1})^{(j+1)}(CF_2^{-1})F_4\Big] = 0$$

and

$$\phi^{(Q)}\Big[(-1)^{Q}(F_4F_2^{-1})^{(Q-(s+1))}(F_4F_1^{-1})^{s}(F_3F_0^{-1}) + \sum_{i_1=1}^{s}(-1)^{Q-i_1}(F_4F_2^{-1})^{(Q-(s+1))}(F_4F_1^{-1})^{(s+1)-j} + \sum_{i_1=s+1}^{Q-1}(-1)^{Q-i_1}(F_4F_2^{-1})^{(Q-i_1)} + I + \sum_{i_1=Q+1}^{S}(-1)^{2Q+1-i_1}\sum_{j=0}^{S-i_1}(F_4F_2^{-1})^{(S+s-(i_1+j)-1)}(F_4F_1^{-1})^{(j+1)}(CF_2^{-1})\Big]e = 1$$

Compute the values of

$$\Omega_{i_1} = \begin{cases} (-1)^{Q-i_1}(F_4F_2^{-1})^{(Q-(s+1))}(F_4F_1^{-1})^s(F_3F_0^{-1}), & i_1 = 0, \\ (-1)^{Q-i_1}(F_4F_2^{-1})^{(Q-(s+1))}(F_4F_1^{-1})^{((s+1)-i_1)}, & i_1 = 1,2,\ldots,s, \\ (-1)^{Q-i_1}(F_4F_2^{-1})^{(Q-i_1)}, & i_1 = s+1,\ldots,Q-1, \\ I, & i_1 = Q \\ (-1)^{2Q+1-i_1}\sum_{j=0}^{S-i_1}(F_4F_2^{-1})^{(S+s-(i_1+1)-j)}(F_4F_1^{-1})^{(j+1)}(CF_2^{-1}), & i_1 = Q+1, Q+2,\ldots,S. \end{cases}$$

Using $\phi^{(Q)}$ and $\Omega_{i_1}, i_1 = 0,1,\ldots,S,$ calculate the value of $\phi^{(i_1)}, i_1 = 0,1,\ldots,S.$ That is,

$$\phi^{(i_1)} = \phi^{(Q)}\Omega_{i_1}, \quad i_1 = 0,1,\ldots,S.$$

4. System Performance Measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

4.1. Expected Inventory Level

Let η_I denote the excepted inventory level in the steady state. Since $\phi^{(i_1)}$ is the steady state probability vector that there are i_1 items in the inventory with each component represents a particular combination of the number of customers in the waiting area 1, number of customers in the waiting area 2 and the status of the server, $\phi^{(i_1)}e$ gives the probability of i_1 item in the inventory in the steady state.

Hence η_I is given by $\eta_I = \sum_{i_1=1}^{S} i_1\phi^{(i_1)}e$

4.2. Expected Reorder Rate

Let η_R denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s+1$ to s . This may occur in the following two cases:

- the server completes the essential service for a high priority customer
- the server completes a service for a low priority customer

Hence we get

$$\eta_R = \mu_1 \sum_{i_3=1}^{N} \sum_{i_4=0}^{M} \phi^{(s+1,1,i_3,i_4)} + \mu_3 \sum_{i_3=0}^{N} \sum_{i_4=1}^{M} \phi^{(s+1,3,i_3,i_4)}.$$

4.3. Expected Loss Rate for High Priority customers

Let η_{LH} denote the expected loss rate for a high priority customer in the steady state. Any arriving high priority customer finds the waiting area 1 is full and leaves the system without getting service. These customers are considered to be lost. Thus we obtain

$$\eta_{LH} = \sum_{i_4=0}^{M} (K-(N+i_4))\lambda_1 \phi^{(0,0,N,i_4)} + \sum_{i_1=1}^{S} \sum_{i_4=0}^{M} (K-(N+i_4))\lambda_1 \phi^{(i_1,1,N,i_4)}$$

$$+ \sum_{i_1=0}^{S} \sum_{i_4=0}^{M} (K-(N+i_4))\lambda_1 \phi^{(i_1,2,N,i_4)} + \sum_{i_1=1}^{S} \sum_{i_4=1}^{M} (K-(N+i_4))\lambda_1 \phi^{(i_1,3,N,i_4)}$$

4.4. Expected Loss Rate for Low Priority customers

Let η_{LL} denote the expected loss rate for a low priority customer in the steady state. Any arriving low priority customer finds the waiting area 2 is full and leaves the system without getting service. These customers are considered to be lost. Hence η_{LL} is given by

$$\eta_{LL} = \sum_{i_3=0}^{N} (K-(i_3+M))\lambda_2 \phi^{(0,0,i_3,M)} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} (K-(i_3+M))\lambda_2 \phi^{(i_1,1,i_3,M)}$$

$$+ \sum_{i_1=0}^{S} \sum_{i_3=1}^{N} (K-(i_3+M))\lambda_2 \phi^{(i_1,2,i_3,M)} + \sum_{i_1=1}^{S} \sum_{i_3=0}^{N} (K-(i_3+M))\lambda_2 \phi^{(i_1,3,i_3,M)}$$

4.5. Expected Waiting Time for High Priority Customers

Let η_{WH} denote the expected waiting time for high priority customers in the waiting area 1. We get $\eta_{WH} = \dfrac{\eta_{QH}}{\eta_{AH}}$,

where η_{QH} is the expected queue length for high priority customers in the waiting area 1. It is given by

$$\eta_{QH} = \sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(i_1,1,i_3,i_4)}$$

$$+\sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(i_1,2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=1}^{M} i_3\phi^{(i_1,3,i_3,i_4)}$$

and η_{AH} is the effective arrival rate for high priority customers (Ross [21]) in the waiting area 1. Then

$$\eta_{AH} = \sum_{i_1=1}^{S}(K-(i_3+i_4))\lambda_1\phi^{(i_1,0,0,0)} + \sum_{i_3=0}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4))\lambda_1\phi^{(0,0,i_3,i_4)}$$

$$+\sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4))\lambda_1\phi^{(i_1,1,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4))\lambda_1\phi^{(i_1,2,i_3,i_4)}$$

$$+\sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\sum_{i_4=1}^{M}(K-(i_3+i_4))\lambda_1\phi^{(i_1,3,i_3,i_4)}$$

4.6. Expected Waiting Time for Low Priority Customers

Let η_{WL} denote the expected waiting time for low priority customers in the waiting area 2. We get $\eta_{WL} = \dfrac{\eta_{QL}}{\eta_{AL}}$,

where η_{QL} is the expected queue length for low priority customers in the waiting area 2. It is given by

$$\eta_{QL} = \sum_{i_3=0}^{N}\sum_{i_4=1}^{M} i_4\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=1}^{M} i_4\phi^{(i_1,1,i_3,i_4)}$$

$$+\sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=1}^{M} i_3\phi^{(i_1,2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=1}^{M} i_4\phi^{(i_1,3,i_3,i_4)}$$

and η_{AL} is the effective arrival rate for low priority customers in the waiting area 2. Then

$$\eta_{AL} = \sum_{i_1=1}^{S}(K-(i_3+i_4))\lambda_2\phi^{(i_1,0,0,0)} + \sum_{i_3=0}^{N}\sum_{i_4=0}^{M-1}(K-(i_3+i_4))\lambda_2\phi^{(0,0,i_3,i_4)}$$
$$+\sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M-1}(K-(i_3+i_4))\lambda_2\phi^{(i_1,1,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M-1}(K-(i_3+i_4))\lambda_2\phi^{(i_1,2,i_3,i_4)}$$
$$+\sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=1}^{M-1}(K-(i_3+i_4))\lambda_2\phi^{(i_1,3,i_3,i_4)}$$

4.7 Probability that Server is Busy

Let η_{SB} denote the probability that server is busy is given by

$$\eta_{SB} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\phi^{(i_1,1,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\phi^{(i_1,2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=1}^{M}\phi^{(i_1,3,i_3,i_4)}$$

4.8 Probability that Server is Idle

Let η_{SI} denote the probability that server is idle is given by

$$\eta_{SI} = \sum_{i_1=1}^{S}\phi^{(i_1,0,0,0)} + \sum_{i_3=0}^{N}\sum_{i_4=0}^{M}\phi^{(0,0,i_3,i_4)}$$

5. Expected Total Cost Rate

We assume various cost elements associated with different system performance measures are given as follows:

c_h – Inventory carrying cost per unit per unit time

c_s – Setup cost perorder

c_{lh} – Cost per high priority customer lost

c_{ll} – Cost per low priority customer lost

c_{wh} – Waiting cost of a high priority customer per unit time

c_{wl} – Waiting cost of a low priority customer per unit time

We construct the function for the expected total cost per unit time as follows:

$$TC(S,s,N,M) = c_h\eta_I + c_s\eta_R + c_{lh}\eta_{LH} + c_{ll}\eta_{LL} + c_{wh}\eta_{WH} + c_{wl}\eta_{WL}$$

where η 's are as given in the above measures of system performance.

6. Concluding Remarks

The stochastic model discussed here is useful in studying the analysis of two queues in parallel with mixed priority service and a finite population. The joint probability distribution of the inventory level, status of the server, number of high priority customers in the waiting hall 1 and number of low priority customers in the waiting hall 2 is derived in the steady state. Various system performance measures and the long-run total expected cost rate are derived. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times follow PH- distributions.

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