Optimum Simple Step-Stress Test with Competing Risks for Failure using Khamis-Higgins model under Type-I Censoring

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Abstract

In this paper we consider the simple step-stress accelerated life test (ALT) under type-I censoring using Khamis-Higgins model (an alternative to the Weibull cumulative exoposure model for the step-stress accelerated life testing) with competing causes of failure. The Khamis-Higgins model is based on time-transformation of the exponential model. The life distribution of each failure cause, which is independent of other, is assumed to be Weibull with log of characteristic life being a linear function of the stress level. Optimum plan for the time-censored step-stress ALT is obtained which minimize the sum over all failure causes of asymptotic variances of the maximum likelihood estimators of the log characteristics life at design stress. The inferential procedures involving design parameters are also studied.

Keywords : Accelerated life test, Step-stress test, Fisher information , Maximum likelihood, Asymptotic variance, Competing causes of failure.

1. Introduction

Engineers and management spend time and money in assessing new designs, identifying causes of failure and trying to eliminate them so that the product produced is acceptable to the consumer. Due to the longer life times of the products, accelerated life testing is used to determine the reliability of the products. In accelerated life testing, the products are tested at high stress conditions and the results are used to draw inferences about the product at the normal operating condition.

To improve the reliability of a product, it is important to identify different causes of failure, collect and analyze failure data obtained at various stages of the life testing experiment. Further, when there are competing causes of failure, the life of a product depends on the characteristics of different failure model. It is therefore, important to consider experiments where specimens are subject to multi-failure model. Several authors have considered the problem of time-censored (see [11]) step-stress ALTs when the product has only one cause of failure, see for example,[1], [2], [3], [4], [6], [12], [17], [18]. [14] obtained maximum likelihood estimators (MLEs) for the parameters of Weibull distribution under the inverse power law using the breakdown time data of electrical

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insulation. [16] proposed a model based on shock models and wear processes, and obtained a non-parametric estimator for the life distribution at use condition. [5] considered a partially accelerated life testing in which if a test unit does not fail by a specified time at a design condition, it is switched to a higher level of stress.

Several papers have been written on analyzing ALT when more than one of the failure causes is present. Here failures can occur from any one of p statistically independent causes. Sample data consist of a failure time and the cause of failure for each item. Assuming that for a given stress V, the lifetime under each failure cause follows independent lognormal distribution with parameters $\mu_k(V) = \alpha_k + \beta_k V$ and σ_k^2, independent of V, k = 1, ..., p, [13] obtained graphical estimates of α_k and β_k when there is no censoring. [9,10] described the analysis of ALTs of series systems with Weibull component failure times for the cases of a common and different shape parameters. ([15], Chapter 7) has discussed the analysis of ALTs under competing causes of failure and gives examples of products which have multiple causes of failure, e.g., semiconductor devices, ball bearing assemblies, and insulation systems.[7] discussed inferential problem for a step-stress model with competing risks for failure from the exponential distribution under time constraint.

A special case of the accelerated life-testing, known as step-stress testing, allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the experiment for maximal flexibility and adjustability. The step-stress ALTs are widely used in; for example, life testing of diodes [4], cable insulation [14], and insulating fluid [12]. They have the advantage of yielding more failure data in a limited time without necessarily using a high stress to all fest units and the asymptotic theory is a better approximation when there are many failures ([15], Chapter 10).

In this paper we have used Weibull cumulative exposure model [8] in step-stress accelerated life testing that we call the transformed exponential model. This paper presents optimum simple step-stress model (i.e. two stress levels) under Type-I censoring with competing causes of failure. The life distribution of each failure cause, which is s-independent of the others, follows Weibull distribution with a median that is a log linear function of the stress, and a cumulative exposure model is assumed. The sum overall failure causes of the asymptotic variances of the MLE, of the log median lives at design stress is used as an optimality criterion, and the optimal stress change time in the time-step stress ALT. Inferential procedures involving design parameters are also discussed.

Notations

x_0, x_1, x_2	Stress (design, low, high)
ξ_j	Extrapolation amount $\xi = \dfrac{(x_1 - x_0)}{(x_2 - x_1)}$
n	number of test units.
n_i	number of failed units at stress x_i, i = 1, 2,

n_c	number of units censored $n_c = n - n_1 - n_2$
λ_{ij}, θ_{ij}	median life, failure rate under failure causes j at stress x_i, i = 0, 1, 2; j = 1, 2, ; $\theta_{ij} = \frac{1}{\lambda_{ij}}$
T	Censoring time
τ	Stress change point
τ^*	Optimal time of changing stress

2. Model and assumptions

The model and assumptions used in this article are:

1. Each unit has two statistically independent potential failure times corresponding to two causes of failure.

2. Failure time of a unit is the smaller of its two potential failure times.

3. Two stress levels x_1 and x_2 ($x_1 < x_2$) are used.

4. For any level of stress, the lifetime under each failure cause follows a Weibull distribution.

$$G_j(w) = 1 - \exp\left\{\frac{-w^{\delta}}{\theta}\right\},\ 0 \le w < \infty,\ j=1,2$$

5. The median level of p = 1, 2 potential failure times are log linear functions of stress:

$$\log \theta_{ij}^{\frac{1}{\delta}} = \alpha_j + \beta_j x_i;\ i = 0, 1, 2;\ j = 1, 2. \quad (1)$$

   α_j, β_j are unknown parameters depending on the nature of the product and the test method.

6. For each failure cause, a cumulative exposure model is assumed. That is, the remaining life of a test unit under each failure cause depends only on the exposure it has seen and does not remember how the exposure was accumulated [12].

2.1 Procedure

1. Initially n test units are placed on low stress x_1 and run until time τ ($0 < \tau < T$), when the stress is changed to x_2 and the test is continued until a pre- assigned censoring time T.

2. Failure times and failure causes of test units are jointly observed continuously.

From the cumulative exposure model and Weibull distributed life assumptions, the cdf of risk factor j = 1, 2 under simple time step-stress test is the Khamis-Higgins model :

$$G_j(w) = \begin{cases} 1 - \exp\left\{ \dfrac{-w^\delta}{\theta_{ij}} \right\} & \text{if } 0 < w < \tau \\ 1\text{-}\exp\left\{ -\dfrac{(w^\delta - \tau^\delta)}{\theta_{ij}} - \dfrac{\tau^\delta}{\theta_{ij}} \right\} & \text{if } \tau \le w < \infty \end{cases},$$

for j = 1, 2 , and the corresponding probability density function (pdf) of W_j is given by

$$g_j(w) = g_j(w; \theta_{ij}, \theta_{ij})$$

$$= \begin{cases} \dfrac{\delta w^{\delta-1}}{\theta_{ij}} e^{-\frac{w^\delta}{\theta_{ij}}} & \text{if } 0 < w < \tau \\ \dfrac{\delta w^{\delta-1}}{\theta_{ij}} e^{\left\{ -\frac{(w^\delta - \tau^\delta)}{\theta_{ij}} - \frac{\tau^\delta}{\theta_{ij}} \right\}} & \text{if } \tau \le w < \infty \end{cases},$$

for j = 1, 2. Since we will observe only the smaller of W_1 and W_2, let W = min $\{w_1, w_2\}$ denote the overall failure time of a test unit then, its cdf and pdf are obtained to be

$$F(w) = 1 - (1 - G_1(w)(1 - G_2(w))$$

$$= \begin{cases} 1 - \exp\left\{ -\left(\dfrac{1}{\theta_{11}} + \dfrac{1}{\theta_{12}} \right) w^\delta \right\} & \text{if } 0 < w < \tau, \\ 1 - \exp\left\{ -(w^\delta - \tau^\delta)\left(\dfrac{1}{\theta_{21}} + \dfrac{1}{\theta_{22}} \right) - \tau^\delta \left(\dfrac{1}{\theta_{11}} + \dfrac{1}{\theta_{12}} \right) \right\} & \text{if } \tau \le w < \infty \end{cases} \quad (2)$$

$$f(w) = f(w', \theta)$$

$$= \begin{cases} \delta w^{\delta-1}\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)\exp\left\{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)w^{\delta}\right\} & \text{if } 0 < w < \tau \\ \delta w^{\delta-1}\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)\exp\left\{-\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)(w^{\delta}-\tau^{\delta})-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)\tau^{\delta}\right\} & \text{if } \tau \le w < \infty , \end{cases} \tag{3}$$

respectively, where $\theta = (\theta_1, \theta_2)$ with $\theta_i = (\theta_{i1}, \theta_{i2})$ for i = 1, 2. Furthermore let C denote the indicator for the cause of failure. Then, under our assumptions, the joint pdf of (W, C) is given by

$$f_{w,c}(w, j) = g_j(t)(1 - G_j(t))$$

$$= \begin{cases} \frac{\delta w^{\delta-1}}{\theta_{ij}}\exp\left\{-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)w^{\delta}\right\} & \text{if } 0 < w < \tau \\ \frac{\delta w^{\delta-1}}{\theta_{ij}}\exp\left\{-\left(\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}}\right)(w^{\delta}-\tau^{\delta})-\left(\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}\right)\tau^{\delta}\right\} & \text{if } \tau \le w < \infty , \end{cases} \tag{4}$$

for j, j′ = 1, 2 and j′ ≠ j. We also denote the relative risk imposed on a test unit before τ due to risk factor j by

$$\pi_{1j} = \Pr[C = j \mid 0 < W < \tau] = \frac{\theta_{ij}^{-1}}{\theta_{11}^{-1} + \theta_{12}^{-1}}, \; j = 1, 2 . \tag{5}$$

Similarly, the relative risk after τ due to the factor j is denoted by

$$\pi_{2j} = \Pr[C = j \mid W \ge \tau] = \frac{\theta_{2j}^{-1}}{\theta_{21}^{-1} + \theta_{22}^{-1}}, \; j = 1, 2. \tag{6}$$

They are simply the proportion of failure rates in the given time frame. One can then easily see from (4)-(6), that W and C are independent given the time frame in which a failure has occurred.

3. Optimum simple step-stress plan when the shape parameter is known

3.1 Log Likelihood

Let us now define

n_{1j} = the number of units that fail before τ due to the risk factor j,

n_{2j} = the number of units that fail after τ due to the risk factor j.

Under the assumption of the cumulative exposure model, we formulate the likelihood function of θ based on the Type-I sample is

$$L(\theta)=\prod_{i=1}^{n_{11}}\left[\frac{\delta w_i^{\delta-1}}{\theta_{11}}e^{-w_i^{\delta}/\theta_{1\cdot}}\right]\prod_{i=1}^{n_{12}}\left[\frac{\delta w_i^{\delta-1}}{\theta_{12}}e^{-w_i^{\delta}/\theta_{i\cdot}}\right]$$
$$\prod_{i=1}^{n_{21}}\left[\frac{\delta w_i^{\delta-1}}{\theta_{21}}e^{-\frac{w_i^{\delta}}{\theta_{2\cdot}}-\frac{\tau^{\delta}}{\theta_{1\cdot}}}\right]\prod_{i=1}^{n_{22}}\left[\frac{\delta w_i^{\delta-1}}{\theta_{12}}e^{-\frac{w_i^{\delta}-\tau^{\delta}}{\theta_{2\cdot}}-\frac{\tau^{\delta}}{\theta_{1\cdot}}}\right]e^{-\left\{\frac{n_c(T^{\delta}-\tau^{\delta})}{\theta_{2\cdot}}+\frac{n_c\tau^{\delta}}{\theta_{1\cdot}}\right\}}, \tag{7}$$

where $\frac{1}{\theta_{1\cdot}}=\frac{1}{\theta_{11}}+\frac{1}{\theta_{12}}$,

$$\frac{1}{\theta_{2\cdot}}=\frac{1}{\theta_{21}}+\frac{1}{\theta_{22}},$$

$$n_{1\cdot}=n_{11}+n_{12},$$

$$n_{2\cdot}=n_{21}+n_{22},$$

$n=n_{1\cdot}+n_{2\cdot}+n_c$, n is fixed and known.

Thus, the log-likelihood, L, of a Type I censored observation is:

$$\begin{aligned}\log L = & -n_{11}\log\theta_{11} + n_{11}\log\delta + (\delta-1)\sum_{i=1}^{n_{11}} \log w_i - \sum_{i=1}^{n_{11}} \frac{w_i^{\delta}}{\theta_{1.}} - n_{12}\log\theta_{12} \\ & + n_{12}\log\delta + (\delta-1)\sum_{i=1}^{n_{12}} \log w_i - \sum_{i=1}^{n_{12}} \frac{w_i^{\delta}}{\theta_{1.}} - n_{21}\log\theta_{21} + n_{21}\log\delta \\ & + (\delta-1)\sum_{i=1}^{n_{21}} \log w_i - \sum_{i=1}^{n_{21}} \left(\frac{w_i^{\delta}-\tau^{\delta}}{\theta_{2.}}\right) - \sum_{i=1}^{n_{21}} \frac{\tau^{\delta}}{\theta_{1.}} - n_{22}\log\theta_{22} \\ & + n_{22}\log\delta + (\delta-1)\sum_{i=1}^{n_{22}} w_i + \sum_{i=1}^{n_{22}} \left(\frac{w_i^{\delta}-\tau^{\delta}}{\theta_{2.}}\right) - \sum_{i=1}^{n_{22}} \frac{\tau^{\delta}}{\theta_{1.}} \\ & - n_c\left\{\frac{T^{\delta}-\tau^{\delta}}{\theta_{2.}} - \frac{\tau^{\delta}}{\theta_{1.}}\right\} .\end{aligned} \tag{8}$$

3.2 Fisher Information matrix

The second partial derivatives of the sample log - likelihood with respect to the model parameters are needed for obtaining Fisher information matrix. For a single observation, the first partial derivatives of log L with respect to α_j and β_j, j = 1, 2, are,

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha_j} = & \ \delta\,\Bigg(-n_{11} + \frac{1}{\theta_{1j}}\sum_{i=1}^{n_{11}} w_i^{\delta} + \frac{1}{\theta_{1j}}\sum_{i=1}^{n_{12}} w_i^{\delta} - n_{21} + \frac{1}{\theta_{2j}}\sum_{i=1}^{n_{21}} w_i^{\delta} - \frac{n_{21}\tau^{\delta}}{\theta_{2j}} \\ & + \frac{n_{21}}{\theta_{1j}}\tau^{\delta} + \frac{1}{\theta_{2j}}\sum_{i=1}^{n_{22}} w_i^{\delta} - \frac{n_{22}\tau^{\delta}}{\theta_{2j}} + \frac{n_{22}}{\theta_{1j}}\tau^{\delta} + \frac{n_c(T^{\delta}-\tau^{\delta})}{\theta_{2j}} + \frac{n_c\tau^{\delta}}{\theta_{1j}} \Bigg) ,\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \beta_j} = & \ \delta\,\Bigg(-n_{11}x_1 + \frac{x_1}{\theta_{1j}}\sum_{i=1}^{n_{11}} w_i^{\delta} + \frac{x_1}{\theta_{1j}}\sum_{i=1}^{n_{12}} w_i^{\delta} - n_{21}x_2 + \frac{x_2}{\theta_{2j}}\sum_{i=1}^{n_{21}} w_i^{\delta} \\ & - \frac{n_{21}\tau^{\delta}}{\theta_{2j}}x_2 + \frac{n_{21}}{\theta_{1j}}x_1\tau^{\delta} + \frac{x_2}{\theta_{2j}}\sum_{i=1}^{n_{22}} w_i^{\delta} - \frac{n_{22}}{\theta_{2j}}\tau^{\delta}x_2 + \frac{n_{22}\tau^{\delta}}{\theta_{1j}}x_1 \\ & + \frac{n_c x_2(T^{\delta}-\tau^{\delta})}{\theta_{2j}} + \frac{n_c\tau^{\delta}}{\theta_{1j}}x_1 \Bigg).\end{aligned} \tag{10}$$

These two expressions, when summed over all the test units and set equal to zero, are called the likelihood equations.

The parameters values that are the solution to these equations are the maximum likelihood estimates.

The second partial derivatives are given below :

$$\frac{\partial^2 \log L}{\partial \alpha_j^2} = \delta^2 \Big(-\frac{1}{\theta_{1j}} \sum_{i=1}^{n_{11}} w_i^{\delta} - \frac{1}{\theta_{1j}} \sum_{i=1}^{n_{12}} w_i^{\delta} - \frac{1}{\theta_{2j}} \sum_{i=1}^{n_{21}} w_i^{\delta} + \frac{n_{21}\tau^{\delta}}{\theta_{2j}} - \frac{n_{21}}{\theta_{1j}} \tau^{\delta} - \frac{1}{\theta_{2j}} \sum_{i=1}^{n_{22}} w_i^{\delta} + \frac{n_{22}}{\theta_{2j}} \tau^{\delta} - \frac{n_{22}}{\theta_{1j}} \tau^{\delta} - \frac{n_c (T^{\delta} - \tau^{\delta})}{\theta_{2j}} - \frac{n_c \tau^{\delta}}{\theta_{1j}} \Big) , \tag{11}$$

$$\frac{\partial^2 \log L}{\partial \beta_j^2} = \delta^2 \Big(-\frac{x_1^2}{\theta_{1j}} \sum_{i=1}^{n_{11}} w_i^{\delta} - \frac{x_1^2}{\theta_{1j}} \sum_{i=1}^{n_{12}} w_i^{\delta} - \frac{x_2^2}{\theta_{2j}} \sum_{i=1}^{n_{21}} w_i^{\delta} + \frac{x_2^2 n_{21} \tau^{\delta}}{\theta_{2j}} - \frac{x_1^2 n_{21}}{\theta_{1j}} \tau^{\delta} - \frac{x_2^2}{\theta_{2j}} \sum_{i=1}^{n_{22}} w_i^{\delta} + \frac{x_2^2 n_{22}}{\theta_{2j}} \tau^{\delta} - \frac{x_1^2 n_{22}}{\theta_{1j}} \tau^{\delta} - \frac{n_c x_2^2 (T^{\delta} - \tau^{\delta})}{\theta_{2j}} - \frac{n_c x_1^2 \tau^{\delta}}{\theta_{1j}} \Big), \tag{12}$$

$$\frac{\partial^2 \log L}{\partial \alpha_j \partial \beta_j} = \delta^2 \Big(-\frac{x_1}{\theta_{1j}} \sum_{i=1}^{n_{11}} w_i^{\delta} - \frac{x_1}{\theta_{1j}} \sum_{i=1}^{n_{12}} w_i^{\delta} - \frac{x_2}{\theta_{2j}} \sum_{i=1}^{n_{21}} w_i^{\delta} + \frac{x_2 n_{21} \tau^{\delta}}{\theta_{2j}} - \frac{x_1 n_{21}}{\theta_{1j}} \tau^{\delta} - \frac{x_2}{\theta_{2j}} \sum_{i=1}^{n_{22}} w_i^{\delta} + \frac{x_2 n_{22}}{\theta_{2j}} \tau^{\delta} - \frac{x_1 n_{11}}{\theta_{1j}} \tau^{\delta} - \frac{n_c x_2 (T^{\delta} - \tau^{\delta})}{\theta_{2j}} - \frac{n_c x_1 \tau^{\delta}}{\theta_{1j}} \Big) , \tag{13}$$

$$\frac{\partial^2 \log L}{\partial \alpha_j \partial \beta_j} = \frac{\partial^2 \log L}{\partial \beta_j \partial \alpha_j} \, . \tag{14}$$

The Fisher information matrix, F, is obtained by taking the negative expectations of the second partial and mixed partial derivatives of $\log L$ with respect to α_j and β_j, j = 1, 2. They are :

$$E\left[\frac{-\partial^2 \log L}{\partial \alpha_j^2} \right] = \frac{n\ \delta^2\ \theta_{1\square}}{\theta_{1j}} A_1(\tau) + \frac{n\ \delta^2\ \theta_{2\square}}{\theta_{2j}} A_2(\tau) = P_j , \tag{15}$$

$$E\left[\frac{-\partial^2 \log L}{\partial \beta_j^2}\right] = n\ \delta^2\ x_1^2 \frac{\theta_{1\square}}{\theta_{1j}} A_1(\tau) + n\ \delta^2\ x_2^2 \frac{\theta_{2\square}}{\theta_{2j}} A_2(\tau) = Q_j, \tag{16}$$

$$E\left[\frac{-\partial^2 \log L}{\partial \alpha_j \partial \beta_j}\right] = n\ \delta^2\ x_1 \frac{\theta_{1\square}}{\theta_{1j}} A_1(\tau) + n\ \delta^2\ x_2 \frac{\theta_{2\square}}{\theta_{2j}} A_2(\tau) = R_j, \tag{17}$$

$$E\left[\frac{-\partial^2 \log L}{\partial \alpha_j \partial \beta_j}\right] = E\left[\frac{-\partial^2 \log L}{\partial \beta_j \partial \alpha_j}\right] = R_j, \tag{18}$$

where

$$A_1(\tau) = 1 - \exp\left\{-\frac{\tau^\delta}{\theta_{1\square}}\right\}, \text{ and } A_2(\tau) = \exp\left\{-\frac{\tau^\delta}{\theta_{1\square}}\right\}\left(1 - \exp\left\{-\frac{(T^\delta - \tau^\delta)}{\theta_{2\square}}\right\}\right).$$

These expectations are calculated from (11) - (13) with the aid of the expectations

$$E\left[\frac{-\partial \log L}{\partial \alpha_j}\right] = 0, \text{ and } E\left[\frac{-\partial \log L}{\partial \beta_j}\right] = 0, \qquad (j = 1, 2).$$

Thus, Fisher information matrix F is ,

$F = \mathrm{Diag}\{ F_1, F_2 \}$,

that is , the matrix whose diagonal 2×2 submatrices F_j , j = 1,2 are :

$$F_j = \begin{bmatrix} P_j & R_j \\ R_j & Q_j \end{bmatrix}.$$

4. Variance of the estimate at design stress

For any plan, the asymptotic covariance matrix of the maximum likelihood estimates α_j , β_j , j = 1,2 is the inverse of the corresponding Fisher information matrix, i.e.,

$$F^{-1} = \begin{bmatrix} \mathrm{Var}[\hat{\alpha}_1] & \mathrm{Cov}[\hat{\alpha}_1, \hat{\beta}_1] & 0 & 0 \\ \mathrm{Cov}[\hat{\beta}_1, \hat{\alpha}_1] & \mathrm{Var}[\hat{\beta}_1] & 0 & 0 \\ 0 & 0 & \mathrm{Var}[\hat{\alpha}_2] & \mathrm{Cov}[\hat{\alpha}_2, \hat{\beta}_2] \\ 0 & 0 & \mathrm{Cov}[\hat{\beta}_2, \hat{\alpha}_2] & \mathrm{Var}[\hat{\beta}_2] \end{bmatrix}$$

Thus, the asymptotic variance of the estimate of log of characteristic life at design stress x_0 is $n\sum_{j=1}^{2} \text{AsVar}\,(\alpha_j + \beta_j x_0)$.

5. Confidence Intervals

Tests of hypotheses for parameters can be obtained either by using the likelihood ratio method or the approximate normality of MLEs in large samples. In the latter case it is most convenient to use the approximation

$$(\hat{\alpha}_j,\ \hat{\beta}_j) \sim N((\alpha_j,\ \beta_j), F^{-1}),\ j = 1, 2.$$

The two-sided 100(1- α^*) approximate confidence intervals for the parameters $\alpha_j,\ \beta_j$, j = 1,2 is

$$\hat{\alpha}_j \pm z_{\alpha^*/2}\sqrt{\hat{\text{Var}}(\hat{\alpha}_j)} \text{ and } \hat{\beta}_j \pm z_{\alpha^*/2}\sqrt{\hat{\text{Var}}(\hat{\beta}_j)},$$

where $z_{\alpha^*/2}$ is the $(1 - \alpha^*/2)^{th}$ quantile of a standard normal distribution, and $\sqrt{\hat{\text{Var}}(\hat{\alpha}_j)}$ and $\sqrt{\hat{\text{Var}}(\hat{\beta}_j)}$, j= 1,2 are obtained by taking square root of the respective diagonal element of F^{-1}. Although this method is quick and easy, one major problem associated with it is that it does not necessarily take the parameter space into account when constructing confidence intervals. There is no built-in procedure to prevent this and as a result, the lower bound (upper bound) of the approximate confidence intervals frequently hits below (above) zero though the parameter can take only positive (negative) values. In order to turn such intervals into sensible ones, the negative (positive) lower (upper) bounds are replaced by zero [3].

6. Testing of Hypotheses

An important inference problem concerning the regression coefficient β_j, j = 1,2 is the test of hypotheses

(i) $H_{01} : \beta_1 = \beta_2 = 0$, against
H_{11} : atleast one of β_j , j=1,2 is distinct from zero.

(ii) $H_{02} : \beta_1 = 0$, against
$H_{12} : \beta_1 < 0$

(iii) $H_{03} : \beta_2 = 0$, against
$H_{13} : \beta_2 < 0$

To test (i), (ii) and (iii), we use the likelihood ratio statistic

$$\Lambda_1 = -2\log\left[\frac{L(\tilde{\alpha}_1,\tilde{\alpha}_2,0,0)}{L(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2)}\right],\ \Lambda_2 = -2\log\left[\frac{L(\tilde{\alpha}_1,\tilde{\alpha}_2,0,\tilde{\beta}_2)}{L(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2)}\right],$$

and $\Lambda_3 = -2\log\left[\frac{L(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}_1,0)}{L(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2)}\right]$,

where $\tilde{\beta}_j$ is the MLE of β_j under null hypothesis H_{0j}, and $\hat{\beta}_j$ is the unrestricted MLE, j= 1,2,3. Large values of Λ_1, Λ_2, Λ_3 provide evidence against their respective null hypothesis. The significant levels can be calculated by using the fact that in large samples Λ_1 is approximately distributed as χ_2^2 ,while Λ_2 and Λ_3 both are approximately distributed as χ_1^2.

7. A Numerical Example

Example 1: The data in Table 1 include 30 simulated observations based on $n = 30, \delta = 2,\ \theta_{11} = 6.08,\ \theta_{12} = 12.03,\ \theta_{21} = 3.05,\ \theta_{22} = 6.02,\ T = 2.8,\ x_0 = 2,$ $x_1 = 3,\ x_2 = 5.$

The optimal value of τ obtained using (2) is $\tau^* = 2.16$. We fit the following model

$$\log\theta_{ij}^{\frac{1}{\delta}} = \alpha_j + \beta_j x_i;\ i = 0, 1, 2;\ j = 1, 2.$$

The MLEs $\hat{\alpha}_1,\ \hat{\alpha}_2,\ \hat{\beta}_1,\ \hat{\beta}_2$ are readily obtained using the NMaximize option of Mathematica 6. The MLEs are

$\hat{\alpha}_1 = 2.67608,\ \hat{\alpha}_2 = 1.67246,\ \hat{\beta}_1 = -0.35588,\ \hat{\beta}_2 = -0.11461.$

The inverse of Fisher observed information matrix $\hat{F}^{-1}$ are given as

$$\hat{F}^{-1} = \begin{bmatrix} 0.4455 & -0.1010 & 0 & 0 \\ -0.1010 & 0.0242 & 0 & 0 \\ 0 & 0 & 0.3375 & -0.08533 \\ 0 & 0 & -0.0853 & 0.02301 \end{bmatrix}.$$

The observed values of F^{-1}, that is $\hat{F}^{-1}$, were determined by substituting the estimated parameters $\hat{\alpha}_1,\ \hat{\alpha}_2,\ \hat{\beta}_1,\ \hat{\beta}_2$ for the true parameters in the asymptotic covariance matrix. To find the standard errors of $\hat{\alpha}_1,\ \hat{\alpha}_2,\ \hat{\beta}_1,\ \hat{\beta}_2$, we take the square root of the diagonal elements of $\hat{F}^{-1}$. 95% confidence intervals for the parameters are

$$1.36783 \le \alpha_1 \le 3.98433, \quad 0.53387 \le \alpha_2 \le 2.81105,$$

$$-0.66059 \le \beta_1 \le -0.05116, \text{and} \ -0.411905 \le \beta_2 \le 0 .$$

Since β_2 can take negative values only, therefore to turn the confidence interval to sensible one, we have replaced positive upper bound by zero (See Section 5).

Table 2 shows the likelihood ratio statistics Λ for tests of various submodels against the full model ($\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2$). The column d.f. indicates the degree of freedom of the Chi-square approximation. The tests based on Λ_1 and Λ_2 lead to the rejection of H_{01} and H_{02} respectively, while that based on Λ_3 leads to acceptance of H_{03}.

8. Sensitivity Analysis

To use an optimum test plan, one needs information about the values of $\theta_{11}, \theta_{12}, \theta_{21}$, and θ_{22} which are usually unknown. Therefore, they have to be approximated from experience, similar data, or preliminary test. Incorrect choice of pre-estimates gives a non optimal test plan and results in poor estimates of the parameters of life distribution at design stress. We have investigated the effect of per-estimates for some selected values of $\theta_{11}, \theta_{12}, \theta_{21}$, and θ_{22} in terms of relative stress change point, τ^* are presented in Table 3.

$$\%\Delta\tau^* = \left(\left| (\tau^* - \tau^0) / \tau^* \right| \right) 100 \ ,$$

where τ^* is the optimal stress change point for the plans obtained with the correctly specified values, and τ^0 is the stress change point for the plans obtained with misspecified values. The sensitivity analysis indicates that because these values have a very small effect on the optimal value τ^*, they are not sensitive. Therefore, the optimum plans proposed are robust, and initial estimates have a small effect on optimal values.

Table 1: Simple step - stress with competing risks of failures simulated data
($n = 30, \delta = 2, \theta_{11} = 6.08, \theta_{12} = 12.03, \theta_{21} = 3.05, \theta_{22} = 6.02, T = 2.8, x_0 = 2,$
$x_1 = 3, x_2 = 5, \tau^* = 2.16$)

Before τ^*		After τ^*	
Failure Time	Failure Cause	Failure Time	Failure Cause
0.4359	2	2.1695	1
0.4515	2	2.1874	2
0.4833	2	2.2159	2
0.5154	1	2.2592	2
0.8941	2	2.2602	1
0.9409	1	2.7544	1
0.9954	1	2.3267	2

1.1158	2	2.3465	1
1.1470	1	2.4216	1
1.4844	2	2.7510	1
1.6796	2		

Table 2: Tests for parameters in the model $\log \theta_{ij}^{\frac{1}{\delta}} = \alpha_j + \beta_j x_i$; i = 0, 1, 2; j = 1, 2.

Model	Λ	**d.f.**	$\chi^2_{(0.05)}$
Full model $(\alpha_1, \alpha_2, \beta_1, \beta_2)$			
$(\alpha_1, \alpha_2, 0, 0)$	5.4346	2	5.99
$(\tilde{\alpha}_1, \tilde{\alpha}_2, 0, \tilde{\beta}_2)$	4.925	1	3.84
$(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, 0)$	0.5094	1	3.84

Table 3 : Effects of incorrect pre-estimates of $\theta_{11} = 6.08$, $\theta_{12} = 12.03$, $\theta_{21} = 3.05$, $\theta_{22} = 6.02$

Percentage Deviation	θ_{11}	θ_{12}	θ_{21}	θ_{22}	$\%\Delta\tau^*$
+1%	6.1408	12.1503	3.0805	6.0802	0.2266
-1%	6.0192	11.9097	3.0195	5.9598	0.2322
+2%	6.2016	12.2706	3.1110	6.1404	0.4478
-2%	5.9584	11.7894	2.9890	5.8996	0.4699
+3%	6.2624	12.3909	3.1415	6.2006	0.6637
-3%	5.8976	11.6690	2.9589	5.8394	0.7138
+4%	6.3232	12.5112	3.1720	6.2608	0.8746
-4%	5.8368	11.5488	2.9280	5.7792	0.9636
+5%	6.3840	12.6315	3.2025	6.3210	1.0804
-5%	5.7760	11.4285	2.8975	5.7190	1.2199

9. Conclusion

In this paper we have studied the optimum time step-stress ALT plan with competing causes of failure using Khamis-Higgins model which is an alternative to the Weibull cumulative exposure model for the step-stress accelerated life testing. The optimum stress change time is obtained by minimizing the sum over all failure causes of asymptotic variances of the MLEs of the log characteristic lives at design stress. Inferential problems pertaining to design parameters have been also studied. Sensitivity analysis results suggest that the optimum plan is robust.

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