Optimum Constant-stress Partially Accelerated Life Tests for the Truncated Logistic Distribution under Time Constraint

P.W. Srivastava* and N. Mittal

Department of Operational Research, Faculty of Mathematical Sciences, University of Delhi (DU), Delhi 110007, India
e-mail: pw_srivastava@yahoo.co.in

Abstract

This paper presents an optimum design of constant-stress partially accelerated life test (PALT) plan in which each item runs either at use or accelerated condition and product life follows Truncated Logistic distribution truncated at point zero under type-I censoring. Truncated distributions arise when sample selection is not possible in some sub-region of sample space. The optimal sample proportion allocated to both normal use condition and accelerated condition for the constant PALT is determined by minimizing the generalized asymptotic variance of MLEs of the acceleration factor and model parameters (D-optimality criterion). The method developed has been illustrated using an example. Sensitivity analysis and comparative study has also been carried out.

Keywords: partially accelerated life test, type-I censoring, fisher information matrix, maximum likelihood estimation

Notations

$\emptyset$	Proportion of sample allocated to accelerated condition
$\bar{\ }$	Implies the complement, e.g, ($\bar{\emptyset} = 1 - \emptyset$)
$\emptyset^*$	Optimum proportion of sample allocated to accelerated condition
$n\bar{\emptyset}$ & $n\emptyset$	Items allocated to use condition and to accelerated condition respectively
n	Total number of test items in a PALT ($n = n\emptyset + n\bar{\emptyset}$)
n_{u_c} & n_{a_c}	Number of censored items at use and at accelerated condition respectively

* corresponding author

n_u & n_a	Number of items failed at use and at accelerated condition respectively
β	Acceleration factor ($\beta > 1$)
T & X	Lifetime of an item at use and at accelerated condition respectively (X = β^{-1}1T)
t_i	Observed lifetime of i^{th} item tested at use condition
x_j	Observed lifetime of jth item tested at accelerated condition
η	Censoring time
μ	Location parameter ($-\infty < \mu < \infty$)
σ	Scale parameter ($\sigma > 0$)
$F\ (u)$	Cumulative distribution function.
$f_1\ (t)$	Probability density function at use condition
$R_1\ (t)$	Reliability function at time't' at use condition
$H_1\ (t)$	Hazard (failure) rate at time't' at use condition
$f_2\ (x)$	Probability density function at accelerated condition
$R_2\ (x)$	Reliability function at time 'x' at accelerated condition
$H_2\ (x)$	Hazard (failure) rate at time 'x' at accelerated condition
P_0	Probability that an item tested only at use condition fails by η, i.e.

$P_0 = \frac{1}{A}((\frac{1}{(1+e^{-\frac{(\eta-\mu)}{\sigma}}})) - (\frac{1}{1+e^{\frac{\mu}{\sigma}}}))$ where $A = \frac{1}{1+e^{-\frac{\mu}{\sigma}}}$

P_1 Probability that an item tested only at accelerated condition fails by η, i.e. $P_1 = \frac{1}{A}((\frac{1}{(1+e^{-\frac{(\beta\eta-\mu)}{\sigma}}})) - (\frac{1}{1+e^{\frac{\mu}{\sigma}}}))$ where $A = \frac{1}{1+e^{-\frac{\mu}{\sigma}}}$

$\delta_{u_i}, \delta_{a_j}$ Indicator functions:

$\delta_{u_i} = 1, T_i \leq \eta, \qquad i = 1,2,\ldots,n_u$

0, otherwise and

$\delta_{a_j} = 1, X_j \leq \eta, \quad j = 1,2, \dots, n_a$

0, otherwise

^ Implies a maximum likelihood estimate

$t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n_u)} \leq \eta$ Ordered failure times at use condition

$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n_a)} \leq \eta$ Ordered failure times at accelerated condition

1. Introduction

Manufacturing designs are improving continuously due to advancement in technology; therefore, it is becoming more and more difficult to obtain information about lifetime of products or materials with high reliability at the time of testing under normal conditions. In such problems, accelerated life tests (ALTs) or partially accelerated life tests (PALTs) are often used to shorten the lives of test items. In an ALT the test items are run only at accelerated condition and in a PALT at both accelerated and use conditions. The major assumption in ALT is that the acceleration factor is known or the mathematical model relating the lifetime of the unit and the stress is known or can be assumed. In some cases, neither acceleration factor nor life-stress relationships are known and cannot even be assumed, that is, ALT data cannot be extrapolated to use conditions. So, in such cases PALT is a more suitable test to be performed for which units are subject to both normal and accelerated conditions (see Abdel – Ghally et al. [1]).

The stress can be applied in various ways, commonly used methods are step-stress and constant-stress (See Nelson [10]). Under step-stress PALT, a test item is first run at normal use condition and, if it does not fail then, it is run at accelerated condition until failure occurs or the observation is censored. But the constant-stress PALT runs each item either at normal use condition or at accelerated condition only. Accelerated test condition includes stresses in the form of temperature, voltage, pressure, vibration, cycling rate, humidity, etc.

The use of a correct life distribution model, especially in the presence of a limited source data – as typically occurs with modern devices having high reliability, helps in presenting the choice of unnecessary and expensive planned replacement.

Some of the commonly used life distribution models in PALT are exponential, weibull, log normal, pareto distribution, burr type – XII and log-logistic.

DeGroot and Goel [6] have considered a PALT and estimated the parameters of the exponential distribution and the acceleration factor using the Bayesian approach. Bai *et al.* [4] have used the maximum likelihood method to estimate the scale parameter and the acceleration factor for the log normally distributed lifetime, using type-I censoring data. Ismail [7] has used maximum likelihood and Bayesian methods for estimating the acceleration factor and the parameters of Pareto distribution of the second kind. Also, Bhattacharyya and Soejoeti [5] have estimated the parameters of the Weibull distribution and acceleration factor using maximum likelihood method. Bai and Chung [3] have considered optimal designs for both step and constant PALTs under type-I censoring. Abdel-Hamid [2] has used the constant stress PALT and estimated the parameters of the burr type-XII distribution and the acceleration factor under Progressive type-II censoring. Yang [11] has indicated that constant-stress accelerated life tests are widely used to save time and money.

However, the failure rate of exponential distribution is constant which is hardly realized in practice. For Weibull distribution the failure rate

$h(x) = \beta^{\gamma}.\gamma.x^{\gamma-1}$ is constant for $\gamma = 1$, and for $\gamma > 1$ and $\gamma < 1$ it leads to two unrealistic situations:

(i) For $\gamma > 1$, $h(x) \to 0$ as $x \to 0$ thereby failing to account for early failures, and $h(x) \to \infty$ as $x \to \infty$ and is therefore unbounded.
(ii) For $\gamma < 1$, $h(x) \to \infty$ as $x \to 0$ there by failing to account for early failures, and $h(x) \to 0$ as $x \to \infty$.

For Burr type-XII distribution the failure rate
$h(x) = (k.\gamma.x^{\gamma-1})/(1 + x^{\gamma})$, $\gamma > 1, x > 0$ is zero for $\gamma > 1$, and for $\gamma = 1$ and $0 < \gamma < 1$ it leads to two unrealistic situations:

(i) For $\gamma = 1$, $h(x) \to 0$ as $x \to \infty$, and $h(x) \to k$ as $x \to 0$ which is constant.

(ii) For $0 < \gamma < 1$, $h(x) \to \infty$ as $x \to 0$ there by failing to account for early failures, and $h(x) \to 0$ as $x \to \infty$.

For Pareto distribution of second kind the failure rate
$h(x) = ((\gamma + 1)\delta) / (1 + \delta.x)$, $x \geq 0$ is constant i.e. $(\gamma + 1)\delta$ as $x \to 0$ and $h(x) \to 0$ as $x \to \infty$.

Further, the failure rate of a lognormal life distribution starts at zero, rises to a peak, and then asymptotically approaches zero which is again unrealistic.

This drawback is overcome by failure rate of truncated logistic distribution truncated at point zero, which is increasing and is more realistically bounded below and above by nonzero finite quantity.

Truncated distributions arise when sample selection is not possible in some sub-region of the sample space. The logistic distribution is inappropriate in modeling lifetime data because the left hand limit of the distribution extends to negative infinity. This could conceivably result in modeling negative times-to-failures. This has necessitated the use of truncated logistic distribution truncated at point zero.

This paper focuses on the maximum likelihood method for estimating the acceleration factor and parameters of the truncated logistic distribution with constant-stress PALT under type-I censored samples. The sample-proportion allocated to both use and accelerated condition for the constant PALT is determined by minimizing the generalized asymptotic variance of MLEs of the acceleration factor and model parameters that is, using D-optimality criterion.

In Section 2 model's basic assumptions are described. Section 3 deals with test procedure of constant PALT. In Section 4 truncated logistic distribution is introduced as failure time model. The maximum likelihood estimates of the model parameters and acceleration factor along with their respective approximate confidence intervals have been obtained in section 5. In Section 6 optimal simple constant-stress PALT is formulated .In Section 7 the method developed has been illustrated using the data simulated from the proposed model with type-I censoring. The sensitivity analysis is also presented. In Section 8 comparative study has been done. Concluding remarks on the method developed are made in Section 9.

2. Basic assumptions

(a) The lifetime of an item tested either at use condition or at accelerated condition follows truncated logistic distribution.

(b) The lifetimes of test units are independent and identically distributed random variables.

(c) The lifetimes $T_i, i = 1, 2 \ldots n\overline{\emptyset}$ of items allocated to use condition and the lifetimes $X_j, j = 1, 2 \ldots n\emptyset$ of items allocated to accelerated condition are mutually statistically independent.

(d) Lifetime of an item at accelerated condition is $X = \beta^{-1}T, (\beta > 1)$.

3. Test procedure

(i) Out of total 'n' items, ' $n\overline{\emptyset}$ ' items randomly chosen are allocated to use condition and the remaining '$n\emptyset$ ' items are allocated to accelerated condition.

(ii) The test is continued until: a) all test items fail, or b) a prescribed censoring time η, and the test condition should remain the same.

4. Truncated logistic distribution

The cumulative distribution function of Truncated Logistic distribution truncated at point zero is given by:

$$F(u) = \frac{1}{A}((\frac{1}{(1+e^{-\frac{(u-\mu)}{\sigma}}})) - (\frac{1}{1+e^{\frac{\mu}{\sigma}}}))),0 < u < \infty , \sigma > 0, -\infty < \mu < \infty, \quad (1)$$

(see Mood et al. [9])

Its p.d.f., reliability function and hazard function are given by

$$f(u) = e^{-\frac{u-\mu}{\sigma}} / (A\sigma(1 + e^{-\frac{u-\mu}{\sigma}})^2, 0 < u < \infty , \sigma > 0, -\infty < \mu < \infty, \quad (2)$$

$$R(u) = e^{-\frac{u-\mu}{\sigma}} / (A(1 + e^{-\frac{u-\mu}{\sigma}}), 0 < u < \infty , \sigma > 0, -\infty < \mu < \infty, \quad (3)$$

$$h(u) = 1/ (\sigma(1 + e^{-\frac{u-\mu}{\sigma}}), 0 < u < \infty , \sigma > 0, -\infty < \mu < \infty, \text{ respectively.} \quad (4)$$

The hazard function in (4) is an increasing function of u, and is bounded by $1/\sigma\,(1 + e^{\mu/\sigma})$ and $1/\,\sigma$.

Thus, the p.d.f of an item tested at use condition and at accelerated condition is given by: $f_T(t) = e^{-\frac{u-\mu}{\sigma}} / (A\sigma(1 + e^{-\frac{u-\mu}{\sigma}})^2, 0 < t < \infty , \sigma > 0, -\infty < \mu < \infty,$ (5)

and

$f_X(x) = (\beta . e^{-\frac{u-\mu}{\sigma}})/ (A\sigma(1 + e^{-\frac{\beta.x-\mu}{\sigma}})^2, 0 < x < \infty \ , \ \sigma > 0 \ , \ -\infty < \mu < \ , \ \beta > 1$ respectively. (6)

The reliability function of an item tested at use and at accelerated condition is given by:

$$R_t(T) = e^{-\frac{t-\mu}{\sigma}} / (A(1 + e^{-\frac{t-\mu}{\sigma}}), 0 < t < \infty , \sigma > 0, -\infty < \mu < \infty, \quad (7)$$

and

$$R_x(x) = e^{-\frac{\beta.x-\mu}{\sigma}} / (A(1 + e^{-\frac{\beta.x-\mu}{\sigma}}), 0 < x < \infty , \sigma > 0, -\infty < \mu < \infty, \beta > 1 \quad (8)$$

respectively.

5. Model Formulation

Maximum likelihood method has been used to estimate the model parameters μ, σ, and acceleration factor, β, from the test data .

The likelihood function based on $(t_1; \delta_{u_1}, \dots, t_{n\bar{\emptyset}}; \delta_{u_{n\bar{\emptyset}}}, x_1; \delta_{a_1}, \dots, x_{n\emptyset}; \delta_{a_{n\emptyset}}$ is

$$L(\mu, \sigma, \beta) = \prod_{i=1}^{n\bar{\emptyset}} L_{u_i}(\mu, \sigma; t_i, \delta_{u_i}) \prod_{j=1}^{n\emptyset} L_{a_j}(\mu, \sigma, \beta; x_j, \delta_{a_j}) \quad (9)$$

where,

$$L_{u_i}(\mu, \sigma; t_i, \delta_{u_i}) = (\frac{e^{-\frac{t_i-\mu}{\sigma}}}{(A\sigma(1+e^{-\frac{t_i-\mu}{\sigma}})^2}))\delta_{u_i}(\frac{e^{-\frac{\eta-\mu}{\sigma}}}{(A(1+e^{-\frac{\eta-\mu}{\sigma}})^2})))\bar{\delta}_{u_i} \quad (10)$$

$$L_{a_j}\left(\mu,\sigma,\beta;x_j,\delta_{a_j}\right)=((\frac{\beta.e^{-\frac{\beta.x_j-\mu}{\sigma}}}{(A\sigma(1+e^{-\frac{\beta.x_j-\mu}{\sigma}})^2}))\delta_{a_j}(\frac{e^{-\frac{\beta.\eta-\mu}{\sigma}}}{(A(1+e^{-\frac{\beta.\eta-\mu}{\sigma}})^2})))\bar{\delta}_{a_j} \qquad (11)$$

and $\bar{\delta}_{u_i}=1-\ \delta_{u_i}, \bar{\delta}_{a_j}-\ 1-\ \delta_{a_j}$.

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. The first order partial derivatives of log-likelihood function of the i^{th} unit with respect to β, μ, and σ are given by:

$$\frac{\partial L_{u_i}}{\partial\beta}=0,$$

$$\frac{\partial L_{u_i}}{\partial\mu}=\frac{1}{\sigma}-(\frac{2\delta_{u_i}e^{-\frac{t_i-\mu}{\sigma}}}{\sigma\left(1+e^{-\frac{t_i-\mu}{\sigma}}\right)}-\frac{e^{\frac{-\mu}{\sigma}}}{\left(\sigma\left(1+e^{\frac{-\mu}{\sigma}}\right)\right)}-\frac{2\bar{\delta}_{u_i}e^{-\frac{\eta-\mu}{\sigma}}}{\sigma\left(1+e^{-\frac{\eta-\mu}{\sigma}}\right)},$$

$$\frac{\partial L_{u_i}}{\partial\sigma}=-(\frac{\bar{\delta}_{u_i}(\eta-\mu)e^{-\frac{\eta-\mu}{\sigma}}}{\sigma^2\left(1+e^{-\frac{\eta-\mu}{\sigma}}\right)}+\frac{\mu e^{\frac{-\mu}{\sigma}}}{\left(\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)\right)}-\frac{2\delta_{u_i}(t_i-\mu)e^{-\frac{t_i-\mu}{\sigma}}}{\sigma^2\left(1+e^{-\frac{t_i-\mu}{\sigma}}\right)}-\frac{\delta_{u_i}}{\sigma}+\frac{\bar{\delta}_{u_i}(\eta-\mu)}{\sigma^2}+\frac{\delta_{u_i}(t_i-\mu)}{\sigma^2},$$

$$\frac{\partial L_{a_j}}{\partial\beta}=-(\frac{\bar{\delta}_{a_j}.\eta}{\sigma}+(\frac{2\delta_{a_j}x_je^{\frac{-(\beta x_j-\mu)}{\sigma}}}{\left(\sigma\left(1+e^{\frac{-(\beta x_j-\mu)}{\sigma}}\right)\right)}+\frac{\delta_{a_j}}{\beta}-\frac{\delta_{a_j}x_j}{\sigma}+\frac{\bar{\delta}_{a_j}\eta e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{(\sigma\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right))},$$

$$\frac{\partial L_{a_j}}{\partial\mu}=-\frac{e^{-\frac{\mu}{\sigma}}}{\sigma(1+e^{\frac{-\mu}{\sigma}})}-(\frac{2\delta_{a_j}e^{\frac{-(\beta x_j-\mu)}{\sigma}}}{\left(\sigma\left(1+e^{\frac{-(\beta x_j-\mu)}{\sigma}}\right)\right)}+\frac{1}{\sigma}-\frac{\bar{\delta}_{a_j}e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{(\sigma\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right))},$$

$$\frac{\partial L_{a_j}}{\partial\sigma}=\frac{\delta_{a_j}.(\beta x_j-\mu)}{\sigma^2}-(\frac{\bar{\delta}_{a_j}(\beta\eta-\mu)e^{-\frac{\beta\eta-\mu}{\sigma}}}{\sigma^2\left(1+e^{-\frac{\beta\eta-\mu}{\sigma}}\right)}-\frac{\delta_{a_j}}{\sigma}-\frac{2\delta_{a_j}(\beta x_j-\mu)e^{-\frac{\beta x_j-\mu}{\sigma}}}{\sigma^2\left(1+e^{-\frac{\beta x_j-\mu}{\sigma}}\right)}+\frac{\bar{\delta}_{a_j}(\beta\eta-\mu)}{\sigma^2}+$$

$$\frac{\mu e^{-\frac{\mu}{\sigma}}}{\sigma^2\left(1+e^{-\frac{\mu}{\sigma}}\right)},$$

On summing these partial derivatives and equating them to zero, likelihood equations are obtained. Since, the closed form solutions of above likelihood equations are very hard to obtain, so further numerical treatment is required to obtain the MLEs of β , μ and σ.

5.1. Fisher information matrix

It is the 3×3 symmetric matrix of expectation of negative second order partial derivatives of the log likelihood function with respect to β, μ, and σ.

$$F(\beta,\mu,\sigma) = \begin{pmatrix} \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\beta^2}] & \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\beta\partial\mu}] & \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\beta\partial\sigma}] \\ \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\mu\partial\beta}] & \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\mu^2}] & \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\mu\partial\sigma}] \\ \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\sigma\partial\beta}] & \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\sigma\partial\mu}] & \sum_{i=1}^{n\bar{\phi}} E[\frac{-\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\sigma^2}] \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\beta^2}] & \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\beta\partial\mu}] & \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\beta\partial\sigma}] \\ \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\mu\partial\beta}] & \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\mu^2}] & \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\mu\partial\sigma}] \\ \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\sigma\partial\beta}] & \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\sigma\partial\mu}] & \sum_{j=1}^{n\phi} E[\frac{-\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\sigma^2}] \end{pmatrix} \quad (15)$$

where the values of these elements are given in Appendix A.

5.2 Confidence intervals

The MLEs $\hat{\beta}$, $\hat{\mu}$ and $\hat{\sigma}$ are approximately normally distributed in large samples, therefore $(\hat{\beta}, \hat{\mu}, \hat{\sigma}) \sim N((\beta, \mu, \sigma), F^{-1})$. The two - sided 100(1$-\alpha$) % approximate confidence interval for the parameter μ is given by $\hat{\mu} \pm z_{\frac{\alpha}{2}}\sqrt{\widehat{var}}(\hat{\mu})$, where $z_{\frac{\alpha}{2}}$ is the $(1-\alpha/2)^{th}$ quantile of a standard normal distribution , and $\sqrt{\widehat{var}}(\hat{\mu})$ is obtained by taking square root of the first diagonal element of F^{-1}. Similarly two-sided $100(1-\alpha)$ % approximate confidence interval for the parameter σ and acceleration factor, β, can be obtained.

Although this method is quick and easy, one major problem associated with it is that it does not necessarily take the parameter space into account when constructing confidence intervals. There is no built-in procedure to prevent this and as a result, the lower bounds of the approximate confidence intervals frequently hit below zero though the parameter can take only positive values. In order to turn such intervals into sensible ones, the negative lower bounds are replaced by zero.

However, Meeker and Escobar [8] have suggested the use of a log transformation to obtain approximate confidence intervals for the parameters that take positive values. Thus, the approximate two sided $100(1-\alpha)$ % confidence intervals for σ and acceleration factor β are

$$(\hat{\beta}e^{\left[\frac{-z_{\frac{\alpha}{2}}\sqrt{\widehat{var}}(\hat{\beta})}{\hat{\beta}}\right]}, \hat{\sigma}e^{\left[\frac{z_{\frac{\alpha}{2}}\sqrt{\widehat{var}}(\hat{\beta})}{\hat{\beta}}\right]} \;\&\; (\hat{\beta}e^{\left[\frac{-z_{\frac{\alpha}{2}}\sqrt{\widehat{var}}(\hat{\beta})}{\hat{\beta}}\right]}, \hat{\beta}e^{\left[\frac{z_{\frac{\alpha}{2}}\sqrt{\widehat{var}}(\hat{\beta})}{\hat{\beta}}\right]}$$ respectively.

6. Optimal test plan

The optimal '$\emptyset$' is found using D-optimality criterion which consists in minimizing the *generalized asymptotic variance* of MLEs of the model parameters and the acceleration factor, that is minimizing the reciprocal of the determinant of Fisher information matrix.

NMinimize option of Mathematica 6 which attempts to find optimum solution using such methods as Nelder Mead, Differential Evolution, Simulated Annealing and Random Search has been used to formulate optimal plans.

Table1 shows that optimal sample proportion allocated to accelerated condition obtained using D-optimality criterion for some selected values of β, μ and σ.

Table 1: Value of ϕ* (n=35, η=6)

β	μ	σ	ϕ*
4.2	1.7	0.7	0.500299
4.4	2.7	0.9	0.503924
4.6	3.7	1.1	0.520195
4.8	4.7	1.3	0.547518
5	5.7	1.5	0.571723
5.2	6.7	1.7	0.604008
5.4	7.7	1.9	0.485752

7. Numerical examples

Assuming $n = 35, \beta = 5, \mu = 5.7$, $\sigma = 1.5, \eta = 6$, the optimal value of 'ϕ' is given by ϕ^* = 0.571723 .

The data in Table 2 gives 35 simulated observations based on data $n = 35$, $\beta = 5$, $\mu = 5.7$, $\sigma = 1.5, \eta = 6$ and ϕ^* = 0.571723. Thus, the total number of units tested at use condition n_u = 15 and at accelerated condition n_a = 20. The MLEs of model parameters and acceleration factor μ, σ and β obtained by using NMaximize option of Mathematica 6 are: $\hat{\mu} = 6.19755,$ $\hat{\sigma} = 1.44937$ and $\hat{\beta} = 5.62781.$

The inverse of observed Fisher information matrix $\hat{F}^{-1}$ is given as:

$$\hat{F}^{-1} = \begin{pmatrix} 6.08501 & 6.17171 & 1.8162 \\ 6.17171 & 6.53573 & 1.85213 \\ 1.8162 & 1.85213 & 0.616762 \end{pmatrix}.$$

The estimated variance of estimates of $\hat{\beta}$, $\hat{\mu}$ and $\hat{\sigma}$ are: $\widehat{var}(\hat{\beta}) = 6.08501$, $\widehat{var}(\hat{\mu}) = 6.53573$, $\widehat{var}(\hat{\sigma}) = 0.616762.$

Table 2: Simple constant-stress simulated data
(n=35, β=5, μ=5.7, σ=1.5, η=6 and ϕ* = 0.571723, n_{a_c} = 0 and, n_{u_c} = 8)

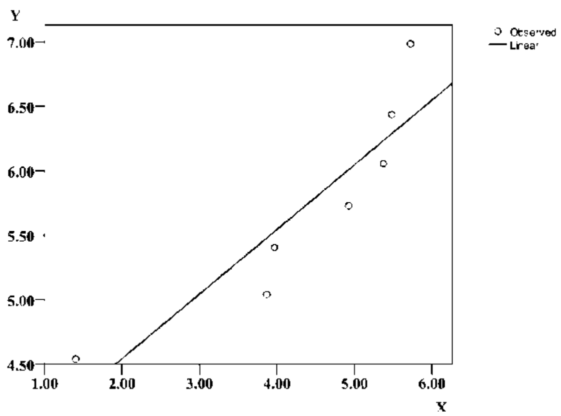
Constant-stress	Failure times
Use condition	4.92484, 5.48164, 1.40047, 5.37267, 3.96824, 5.72282, 3.86573.
Accelerated condition	1.07494, 0.787481, 1.07114, 1.05095, 1.14061, 0.491348, 1.44553, 1.33129, 2.22073, 0.671395, 0.81419, 1.32561, 1.18979, 0.763808, 1.47626, 0.2372, 0.870767, 1.0147, 1.74247, 1.74384.

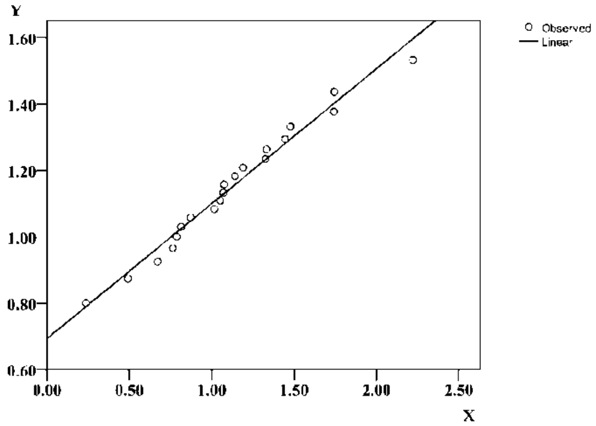
To find the standard errors of $\hat{\beta}$, $\hat{\mu}$ and $\hat{\sigma}$, we take the square root of the diagonal elements of $\hat{F}^{-1}$, 95% confidence intervals for the acceleration factor and model parameters using $\hat{\beta} \times e^{\pm z_{0.025}\frac{\widehat{var}(\hat{\beta})}{\hat{\beta}}}$, $\hat{\mu} \pm z_{0.025}\widehat{var}(\hat{\mu})$ and $\hat{\sigma} \times e^{\pm z_{0.025}\frac{\widehat{var}(\hat{\sigma})}{\hat{\sigma}}}$ are respectively $2.3836 \leq \beta \leq 13.2875$, $1.18679 \leq \mu \leq 11.2083$ and $0.501126 \leq \sigma \leq 4.19191$.

7.1 Graphical goodness of fit

Figure 1 and Figure 2 shows the constant-stress truncated logistic probability plot at use condition and at accelerated condition respectively with Type-I censoring using generalized asymptotic variance approach. These figures are being added to verify the results obtained, through graphical approach also. The failure times at use condition and at accelerated condition in Table 2 are both arranged separately in increasing order and are ranked from $i = 1,2,\ldots,k,\ldots,n$ and $j = 1,2,\ldots,k,\ldots,n$ respectively. The plot is obtained by taking these ranked failure times on X-axis, and their corresponding ordinate values, viz.,

$\mu - \sigma ln\left(\left(\frac{A_i}{n_u+1}+\frac{1}{1+e^{\frac{\mu}{\sigma}}}\right)^{-1} - 1\right)$ at use condition and $\frac{\mu}{\beta} - \frac{\sigma}{\beta} ln\left(\left(\frac{A_j}{n_a+1}+\frac{1}{1+e^{\frac{\mu}{\sigma}}}\right)^{-1} - 1\right)$ at accelerated condition, where $\emptyset^* = 0.571723$, on Y-axis. The ordinate values are obtained by using (1). The plotted points tend to follow straight line, which is substantiated by fitting straight line to these points resulting in high value of coefficient of determination $r^2 = 0.818$ at use condition and $r^2 = 0.978$ at accelerated condition.





The constant-stress truncated logistic distribution therefore appears to describe the data adequately.

7.2 Sensitivity analysis

To formulate optimum test plan, the information about acceleration factor β and parameters of the model μ, and σ is needed as, incorrect choice of these, results in poor estimates of the quantile. We have studied the effects of incorrect pre-estimates of β, μ and σ in terms of the relative increase of generalized asymptotic variance of log of quantile at Relative Variance = |(GeAsVar* - GeAsVar^0)/GeAsVar*|×100 where GeAsVar* is the generalized asymptotic variance for the plan obtained with the correctly specified values, and GeAsVar^0 is the generalized asymptotic variance for the plan obtained with incorrect-specified values. We have found that if they are not too far from the true values the increase is small, as shown in table 3. Therefore, the proposed optimum plan is robust.

Table 3: Effects of incorrect pre-estimates of
β, μ and σ (n=35, β=5, μ=57, σ=1.5, η=6, GeAsVar*=0.0100562, ϕ*=0.571723)

Percentage Deviation	**β**	**μ**	**σ**	**Relative Variance**	**Incorrect ϕ***	**Incorrect Variance**
–1%	4.95	5.643	1.485	7.94988	0.57022	0.00925674
+1%	5.05	5.757	1.515	8.61966	0.573249	0.010923
–2%	4.90	5.586	1.470	15.2821	0.568735	0.0085194
+2%	5.10	5.814	1.530	17.9656	0.574802	0.0118628
–3%	4.85	5.529	1.455	22.0447	0.567265	0.00783933
+3%	5.15	5.875	1.545	28.0774	0.576476	0.0128797
–4%	4.80	5.472	1.440	28.282	0.565806	0.0072121
+4%	5.20	5.928	1.560	39.086	0.578003	0.0139868
–5%	4.75	5.415	1.425	34.0349	0.564351	0.00663358
+5%	5.25	5.985	1.575	50.9986	0.579657	0.0151847

8. Comparative Study

In this section, the proposed constant-stress PALT model have been compared with the one designed by Bai and Chung [3] in terms of likelihood functions using the hypothetical failure time data set under constant-stress PALT with type-I censoring given in table 4.

Table 4: Failure time data set under constant-stress PALT with type-I censoring

Constant - stress	Failure times	Number of censored units
Use condition	1.50427, 2.41376, 0.916978, 1.39272, 1.80982, 2.36701, 1.21246, 2.34946, 0.240746, 2.52951, 1.34401, 2.63345, 0.355038, 2.78409, 0.898276	2
Accelerated condition	0.20643, 0.334382, 0.32286, 0.517991, 0.410672, 0.571176, 0.378891, 0.482979, 0.338261, 0.299476, 0.616791, 0.194214, 0.278806, 0.230267, 0.155213, 0.497737, 0.514582, 0.34768	0

Table 5. Comparative study of constant-stress PALT models

ALT Model	**Log-likelihood Function**
Proposed Model	–13.221
Bai and Chung [7] model	–25.9753

Table 5 shows that the proposed model performs better than Bai and Chung model for the given data set.

9. Conclusions

In this paper we have obtained an optimum design of constant-stress PALT for the truncated logistic distribution under time constraint using D-optimality criterion. We have also obtained confidence intervals involving acceleration factor and parameters of the model. The effect of incorrect pre-estimates of the acceleration factor and parameters of the model have also been obtained in terms of the percentage of the variance increase for some selected values of the acceleration factor and parameters of the model. We have found that if the mis-specified values are not far from the values the increase is small. Comparative study has also been done with respect to previously studied constant-stress PALT models under type I censoring which shows that proposed model performs better than any other constant-stress PALT models with type I censoring existing in the literature for the given data set.

Appendix A

The elements of Fisher Information matrix are

$$E\left[-\frac{\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\beta^2}\right]=0,$$

$$E\left[-\frac{\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\mu\partial\beta}\right]=0,$$

$$E\left[-\frac{\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\sigma\partial\beta}\right]=0,$$

$$E\left[-\frac{\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\mu^2}\right]=$$

$$-\frac{e^{\frac{-\mu}{\sigma}}}{\left(\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)^2\right)}-\frac{\left(e^{\frac{-\mu}{\sigma}}\right)^2}{\left(\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)\right)}-\frac{2}{\left(3A\sigma^2\left(1+e^{\frac{-(\eta-\mu}{\sigma}}\right)^3\right)}+\frac{1}{\left(A\sigma^2\left(1+e^{\frac{-(\eta-\mu}{\sigma}}\right)^2\right)}+\frac{\left(2\left(e^{\frac{-\mu}{\sigma}}\right)^3\right)}{\left(3\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)^2\right)}+$$

$$\frac{\left(e^{\frac{-(\eta-\mu}{\sigma}}\right)^2}{\left(A\sigma^2\left(1+e^{\frac{-(\eta-\mu}{\sigma}}\right)^3\right)},$$

$$E\left[-\frac{\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\sigma\partial\mu}\right]=ln\frac{\frac{e^{\frac{-(\eta-\mu)}{\sigma}}}{(1+e^{\frac{-(\eta-\mu)}{\sigma}})}}{3A\sigma^2}-ln\frac{\frac{1}{(1+e^{\frac{-\mu}{\sigma}})}}{3A\sigma^2}+\frac{\mu e^{\frac{-\mu}{\sigma}}}{\sigma^3(1+e^{\frac{-\mu}{\sigma}})^2}-\frac{2}{3A\sigma^2(1+e^{\frac{-(\eta-\mu)}{\sigma}})}-$$

$$\frac{1}{3A\sigma^2(1+e^{\frac{-(\eta-\mu)}{\sigma}})^2}-\frac{e^{\frac{-(\eta-\mu)}{\sigma}}}{A\sigma^2(1+e^{\frac{-(\eta-\mu)}{\sigma}})}+\frac{1}{\sigma^2(1+e^{\frac{-\mu}{\sigma}})}+\frac{\eta-\mu}{A\sigma^3(1+e^{\frac{-(\eta-\mu)}{\sigma}})^2}+\frac{5e^{\frac{-\mu}{\sigma}}}{3\sigma^2}+$$

$$\frac{(\eta-\mu)e^{\frac{-(\eta-\mu)^2}{\sigma}}}{A\sigma^3(1+e^{\frac{-(\eta-\mu)}{\sigma}})^3}-\frac{2(\eta-\mu)}{3A\sigma^3(1+e^{\frac{-(\eta-\mu)}{\sigma}})^3}-\frac{2\mu\left(e^{\frac{-\mu}{\sigma}}\right)^3}{3\sigma^3(1+e^{\frac{-\mu}{\sigma}})^2}-\frac{2\left(e^{\frac{-\mu}{\sigma}}\right)^2}{3\sigma^2(1+e^{\frac{-\mu}{\sigma}})}+\frac{\mu\left(e^{\frac{-\mu}{\sigma}}\right)^2}{\sigma^3(1+e^{\frac{-\mu}{\sigma}})},$$

$$E\left[-\frac{\partial^2 lnL_{u_i}(\beta,\mu,\sigma)}{\partial\sigma^2}\right]=\frac{\left(\frac{\eta-\mu}{\sigma^2}\right)^2(e^{\frac{-(\eta-\mu)}{\sigma}})^2}{A(1+e^{\frac{-(\eta-\mu)}{\sigma}})^3}+\frac{e^{\frac{-(\eta-\mu)}{\sigma}}}{A\sigma^2(1+e^{\frac{-(\eta-\mu)}{\sigma}})}-\frac{2e^{\frac{-\mu}{\sigma}}}{\sigma^2}-\frac{1}{\sigma^2}-\frac{\mu^2e^{\frac{-\mu}{\sigma}}}{\sigma^4(1+e^{\frac{-(\eta-\mu)}{\sigma}})^2}+$$

$$\frac{2}{A\sigma^2(1+e^{\frac{-(\eta-\mu)}{\sigma}})}+\frac{2}{A\sigma^5}\int_0^\eta\frac{(t_i-\mu)^2(e^{\frac{-(t_i-\mu)}{\sigma}})^2}{(1+e^{\frac{-(t_i-\mu)}{\sigma}})^4}dt_i,$$

$$E\left[-\frac{\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\beta^2}\right]=\frac{1}{\beta^2}+\frac{\left(\frac{\eta}{\sigma}\right)^2(e^{\frac{-(\beta\eta-\mu)}{\sigma}})^2}{A(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^3}-\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{A\beta^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)}+\frac{2\beta}{A\sigma^3}\int_0^\eta\frac{x_j{}^2(e^{\frac{-(\beta x_j-\mu)}{\sigma}})^2}{(1+e^{\frac{-(\beta x_j-\mu)}{\sigma}})^4}dx_j,$$

$$E\left[-\frac{\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\mu\partial\beta}\right]=\frac{2\eta(e^{\frac{-(\beta\eta-\mu)}{\sigma}})^2}{3A\sigma^2(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^3}-\frac{\eta}{A\sigma^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}+ln\frac{\frac{1}{\left(1+e^{\frac{-\mu}{\sigma}}\right)}}{3A\sigma\beta}+\frac{e^{\frac{-\mu}{\sigma}}}{3\sigma\beta\left(1+e^{\frac{-\mu}{\sigma}}\right)}-$$

$$\frac{\eta\left(e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}{A\sigma^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^3}-\frac{ln\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}}}{3A\sigma\beta}-\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{3A\sigma\beta 1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}},$$

$$E\left[-\frac{\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\sigma\partial\beta}\right]=-\frac{1}{A\sigma\beta\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)}+\frac{\mu}{A\sigma^2\beta\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)}+\frac{\eta\mu}{A\sigma^3\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}-$$

$$\frac{\mu}{3\beta A\sigma^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}+\frac{\mu ln\left(1+e^{\frac{-\mu}{\sigma}}\right)}{3\beta A\sigma^2}+\frac{\mu ln\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)}}{3\beta A\sigma^2}-\frac{\eta(\beta\eta-\mu)\left(e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}{A\sigma^3\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^3}-$$

$$\frac{2\eta\mu}{3A\sigma^3\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^3}+\frac{e^{\frac{-\mu}{\sigma}}}{\sigma\beta}-\frac{\mu e^{\frac{-\mu}{\sigma}}}{3\beta\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)}-\frac{2\beta^2}{A\sigma^4}\int_0^\eta\frac{x_j{}^2(e^{\frac{-(\beta x_j-\mu)}{\sigma}})^2}{(1+e^{\frac{-(\beta x_j-\mu)}{\sigma}})^4}dx_j,$$

$$E\left[-\frac{\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\mu^2}\right]=$$

$$-\frac{e^{\frac{-\mu}{\sigma}}}{\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)^2}-\frac{\left(e^{\frac{-\mu}{\sigma}}\right)^2}{\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)}-\frac{2}{3A\sigma^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^3}+\frac{1}{A\sigma^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}+\frac{2\left(e^{\frac{-\mu}{\sigma}}\right)^3}{3\sigma^2\left(1+e^{\frac{-\mu}{\sigma}}\right)^2}+$$

$$+\frac{\left(e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^2}{A\sigma^2\left(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}}\right)^3},$$

$$E\left[-\frac{\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\sigma\partial\mu}\right]=ln\frac{\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})}}{3A\sigma^2}-ln\frac{\frac{1}{(1+e^{\frac{-\mu}{\sigma}})}}{3A\sigma^2}+\frac{\mu e^{\frac{-\mu}{\sigma}}}{\sigma^3(1+e^{\frac{-\mu}{\sigma}})^2}-\frac{2}{3A\sigma^2(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})}-$$

$$\frac{1}{3A\sigma^2(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^2}-\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{A\sigma^2(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})}+\frac{1}{\sigma^2(1+e^{\frac{-\mu}{\sigma}})}+\frac{\beta\eta-\mu}{A\sigma^3(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^2}+\frac{\mu\left(e^{\frac{-\mu}{\sigma}}\right)^2}{\sigma^3(1+e^{\frac{-\mu}{\sigma}})}+$$

$$\frac{(\beta\eta-\mu)e^{\frac{-(\beta\eta-\mu)}{\sigma}^2}}{A\sigma^3(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^3}-\frac{2(\beta\eta-\mu)}{3A\sigma^3(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^3}-\frac{5\mu e^{\frac{-\mu}{\sigma}}}{3\sigma^2}-\frac{2\mu\left(e^{\frac{-\mu}{\sigma}}\right)^3}{3\sigma^3(1+e^{\frac{-\mu}{\sigma}})^2}-\frac{2\left(e^{\frac{-\mu}{\sigma}}\right)^2}{3\sigma^3(1+e^{\frac{-\mu}{\sigma}})},$$

$$E\left[-\frac{\partial^2 lnL_{a_j}(\beta,\mu,\sigma)}{\partial\sigma^2}\right]=\frac{\left(\frac{\beta\eta-\mu}{\sigma^2}\right)^2(e^{\frac{-(\beta\eta-\mu)}{\sigma}})^2}{A(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})^3}+\frac{e^{\frac{-(\beta\eta-\mu)}{\sigma}}}{A\sigma^2(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})}-\frac{2e^{\frac{-\mu}{\sigma}}}{\sigma^2}-\frac{1}{\sigma^2}-\frac{\mu^2 e^{\frac{-\mu}{\sigma}}}{\sigma^4(1+e^{\frac{-(\eta-\mu)}{\sigma}})^2}+$$

$$\frac{2}{A\sigma^2(1+e^{\frac{-(\beta\eta-\mu)}{\sigma}})}+\frac{2\beta}{A\sigma^5}\int_0^\eta\frac{(\beta x_j-\mu)^2(e^{\frac{-(\beta x_j-\mu)}{\sigma}})^2}{(1+e^{\frac{-(\beta x_j-\mu)}{\sigma}})^4}dx_j,$$

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