Time Dependent Solution of an M/M/1/N Queuing Model Subjected to Varying Catastrophic Intensity with Restoration (A different case)

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Abstract

In this paper, we have studied a finite capacity Markovian queueing model subjected to varying catastrophic intensity and restoration. So far it is assumed that during the restoration time no arrival is allowed to occur. But it is not always the case. The necessary amendment is incorporated in this paper in the form that the customers may arrive during the restoration time. The transient solution has been obtained recursively. Steady state solution and various measures of performance have also been provided. Further, some particular cases and graphical illustrations of the queuing model have also been derived and discussed.

Keywords: Varying catastrophic Intensity, Restoration time, Markovian queueing system, Steady state solution.

1. Introduction

Catastrophic queueing models have been studied during the last three decades. A large number of research papers are available on population processes, computer communications and biosciences under the influence of catastrophes, for instance see, Swift [9], Kyriakidis [8], Brockwell [1, 2], B. Krishna Kumar [7], Di. Crescenzo [4]. In the above mentioned research work the authors has assumed that the random occurrence of catastrophes destroys all the customers in a queueing system. But this assumption does not hold well in many practical situations. So, the necessary amendment is incorporated by Jain and Bura [5, 6] in the form of varying catastrophic intensity to destroy a finite number of customers at a time. The concept of restoration along with varying catastrophic intensity is considered by Gulab Singh Bura [3]. So far it is assumed that during the restoration time no customer is allowed to occur. In the present paper we consider a different case in which the customers may arrive during the restoration time. The introduction of this concept makes the system more generalized from practical point of view. The concept of varying catastrophic intensity has numerous applications in wide variety of areas particularly in computer communication, agriculture and biosciences etc.

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2. Queuing Model:-

The queuing model investigated in this paper is based on the following assumptions:-

(i) The customers arrive in the system one by one in accordance with a Poisson process at a single service station with rate $\lambda > 0$.

(ii) The customers are served one by one at the single channel. The service times are independently identically exponentially distributed with rate $\mu > 0$.

(iii) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ and intensity C_r, $(r = 1, 2, 3, \ldots, N), \sum_{r=1}^{N} C_r = 1$. It depends upon the intensity of the catastrophe that it destroys all the customers or not. If it destroys all the customers, then the system will require some sort of time to re-function in a normal way, which is taken as restoration time. \

(iv) The restoration time is independently, identically and exponentially distributed with parameter β. The customers may arrive during the restoration time.

(v) The queue discipline is first- come- first- served.

(vi) Initially, there are zero customers in the system.

(vii) The capacity of the system is limited to N. i.e., if at any instant there are N customers in the system, then the customers arriving in the duration for which the system remains in state N are not permitted to join the queue and considered lost for the system with probability one.

Define

$P_{00}(t)$ = the probability that there are zero customer in the system at time t without the occurrence of catastrophe.

$Q_{00}(t)$ = the probability that there are zero customer in the system at time t with the occurrence of catastrophe destroying all the customers.

$P_n(t)$ = the probability that there are n customers in the system at time t.

3. Transient Solution

The differential-difference equations governing the system are:

$$P_{00}^{/}(t) = -\lambda P_{00}(t) + \mu P_1(t) + \beta Q_{00}(t) \tag{1}$$

$$Q_{00}^{/}(t) = -(\lambda+\beta)Q_{00}(t) + \xi\left(\sum_{n=1}^{N}\sum_{r=n}^{N} c_r P_n(t)\right) \tag{2}$$

$$P_n^{/}(t) = -(\lambda+\mu+\xi)P_n(t) + \lambda P_{(n-1)}(t) + \mu P_{(n+1)}(t) + \xi\sum_{r=1}^{N-n} c_r\, P_{(n+r)}(t), \quad n = 1, 2, 3, \ldots\ldots, N-1 \tag{3}$$

$$P_N^{/}(t) = -(\mu+\xi)P_N(t) + \lambda P_{(N-1)}(t) \tag{4}$$

Taking, Laplace Transform of equations (1) to (4) w.r.t.'t', we have

$$sP_{00}^{*}(s) = 1 - \lambda P_{00}^{*}(s) + \mu P_1^{*}(s) + \beta Q_{00}^{*}(s) \tag{5}$$

$$sQ_{00}^{*}(s) = -(\lambda+\beta)Q_{00}^{*}(s) + \xi\left(\sum_{n=1}^{N}\sum_{r=n}^{N} c_r P_n^{*}(s)\right) \tag{6}$$

$$sP_n^{*}(s) = -(\lambda+\mu+\xi)P_n^{*}(s) + \lambda P_{(n-1)}^{*}(s) + \mu P_{(n+1)}^{*}(s) + \xi\sum_{r=1}^{N-n} c_r\, P_{(n+r)}^{*}(s) \tag{7}$$

$$sP_N^{*}(s) = -(\mu+\xi)P_N^{*}(s) + \lambda P_{(N-1)}^{*}(s) \tag{8}$$

Where

$$P_n^{*}(s) = \int_0^{\infty} e^{-st} P_n(t)\,dt \qquad \text{and} \qquad P_{00}(0) = 1$$

Solving this set of equations recursively, we have

$$P_n^{*}(s) = \rho^{-N}\left\{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}P_N^{*}(s) \tag{9}$$

$$Q_{00}^{*}(s) = \frac{\xi\rho^{-N}}{s+\lambda+\beta}\sum_{n=1}^{N}\sum_{r=n}^{N}C_r\left\{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}P_N^{*}(s) \tag{10}$$

And

$$P_{00}^{*}(s) = \frac{1}{s+\lambda}\left[1+\rho^{-N}\left\{\begin{array}{l}\left(\frac{\mu(s+\lambda+\beta)+\beta\xi}{s+\lambda+\beta}\right)\left\{\rho+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+1}\ D_i\right\}\\ \frac{\beta\xi}{s+\lambda+\beta}\sum_{n=2}^{N}\sum_{r=n}^{N}C_r\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}\end{array}\right\}P_N^{*}(s)\right] \tag{11}$$

where

$$\rho = \left(\frac{\lambda}{s+\mu+\xi}\right), \quad \eta = \prod_{j=1}^{i}\left(\frac{s+\xi\left(1-\sum_{r_j=1}^{k_j}C_{r_j}\right)}{s+\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}, \qquad [k] \rightarrow \text{An integral function.}$$

$$\prod_{j}^{i} = 1 \ \text{ and } \ \sum_{j}^{i} = 0 \ \text{ for } i < j\ ,\ k_0 = 0,$$

$$A_m = \frac{N-n-(i-m)l_0 + \sum_{a=1}^{i-m-1} l_a - k_m\, l_{(i-m)} - \sum_{b=1}^{m-1}\left(k_{(m-b)} - k_{(m-(b-1))}\right)l_{(i+b-m)}}{l_0 - l_{(i-m)}}$$

$$L_i = \sum_{j=1}^{i}\left(l_{(i-j)} - l_{(i-(j-1))}\right)k_j \quad , \qquad l_j = \begin{cases}0 & \text{if } j = i\\ l_j & \text{if } 1 \le j < i\end{cases} \quad ,$$

$$D_i = \prod_{j=1}^{i}\binom{N-n-L_i-l_{(i-(j-1))}}{l_{(i-j)}-l_{(i-(j-1))}}$$

Using normalization condition, we have

$$P_N^*(s)=\frac{\lambda}{s\rho^{-N}\left[\begin{array}{l}(\mu+\xi)\left\{\rho+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+1-N}\,D_i\right\}\\+\left(\xi\sum\limits_{n=2}^{N}\sum\limits_{r=n}^{N}C_r+(s+\lambda)\sum\limits_{n=1}^{N}\right)\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}\,D_i\right\}\end{array}\right]}\tag{12}$$

Using (12) in (9), (10) and (11) we have

$$P_n^*(s)=\frac{\lambda\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}\,D_i\right\}}{s\left[\begin{array}{l}(\mu+\xi)\left\{\rho+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+1-N}\,D_i\right\}\\+\left(\xi\sum\limits_{n=2}^{N}\sum\limits_{r=n}^{N}C_r+(s+\lambda)\sum\limits_{n=1}^{N}\right)\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}\,D_i\right\}\end{array}\right]}\tag{13}$$

$$Q_{00}^*(s)=\frac{\lambda\xi\sum\limits_{n=1}^{N}\sum\limits_{r=n}^{N}C_r\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}\,D_i\right\}}{s(s+\lambda+\beta)\left[\begin{array}{l}(\mu+\xi)\left\{\rho+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+1-N}\,D_i\right\}\\+\left(\xi\sum\limits_{n=2}^{N}\sum\limits_{r=n}^{N}C_r+(s+\lambda)\sum\limits_{n=1}^{N}\right)\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}\,D_i\right\}\end{array}\right]}\tag{14}$$

$$P_{00}^{*}(s)=\frac{1}{s+\lambda}+\frac{\lambda\left\{\begin{array}{l}\left(\dfrac{\mu(s+\lambda+\beta)+\beta\xi}{(s+\lambda)(s+\lambda+\beta)}\right)\left\{\rho+\displaystyle\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\,\rho^{L_i+1}D_i\right\}\\ \dfrac{\beta\xi}{(s+\lambda)(s+\lambda+\beta)}\displaystyle\sum_{n=2}^{N}\sum_{r=n}^{N}C_r\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\,\rho^{L_i+n}D_i\right\}\end{array}\right\}}{s\left[\begin{array}{l}(\mu+\xi)\left\{\rho+\displaystyle\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\,\rho^{L_i+1-N}D_i\right\}\\ +\left(\xi\displaystyle\sum_{n=2}^{N}\sum_{r=n}^{N}C_r+(s+\lambda)\sum_{n=1}^{N}\right)\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\,\rho^{L_i+n}D_i\right\}\end{array}\right]}\quad(15)$$

After taking Laplace inverse of (13), (14) and (15), we can find all the probabilities.

4. Steady State Solution:

Using the property,

$\lim_{s\to 0} s\,P_n^{*}(s) = P_n$, We have from (13), (14) and (15).

$$P_n=\frac{\rho^n+\displaystyle\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}D_i}{\displaystyle\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\;\rho^{L_i+n}D_i\right\}}$$

$$n=1,2,\ldots\ldots,N \quad (16)$$

$$Q_{00}=\frac{\xi}{(\lambda+\beta)}\sum_{n=1}^{N}\sum_{r=n}^{N}C_r\left(\frac{\rho^n+\displaystyle\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i}{\displaystyle\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}}\right) \tag{17}$$

$$P_{00}=\frac{\left[\begin{array}{l}(\mu(\lambda+\beta)+\beta\xi)\left\{\rho+\displaystyle\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+1}\ D_i\right\}+\\ \beta\xi\displaystyle\sum_{n=2}^{N}\sum_{r=n}^{N}C_r\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}\end{array}\right]}{\lambda(\lambda+\beta)\displaystyle\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}} \tag{18}$$

5. Measure of Effectiveness

The steady state probability distribution for the system size allows us to calculate what are commonly called measures of effectiveness. Two, of immediate interest are the expected number of customers in the system and the expected number of customers in the queue. To derive the foregoing measures, let L_s represents the expected number in the system and L_q represents the expected number in the queue. Thus we have

$$L_s=\sum_{n=1}^{N}n\left\{\frac{\rho^n+\displaystyle\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i}{\displaystyle\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}}\right\}$$

And

$$L_q = \sum_{n=1}^{N}(n-1)\left\{\frac{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i}{\sum\limits_{n=0}^{N}\left\{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\eta\ \rho^{L_i+n}\ D_i\right\}}\right\}$$

6. Important Particular Cases of the model:-

(1) In case the catastrophic intensity follows the uniform distribution then we get:

$$P_n = \frac{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}\ D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum\limits_{n=0}^{N}\left\{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}\ D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}$$

$$Q_{00} = \frac{\xi}{(\lambda+\beta)}\sum_{n=1}^{N}\sum_{r=n}^{N}C_r\left(\frac{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}\ D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum\limits_{n=0}^{N}\left\{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}\ D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}\right)$$

$$P_{00}=\frac{\left[\begin{aligned}&(\mu(\lambda+\beta)+\beta\xi)\left\{\rho+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+1}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}+\\&\beta\xi\sum_{n=2}^{N}\sum_{r=n}^{N}C_r\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}\end{aligned}\right]}{\lambda(\lambda+\beta)\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}$$

$$L_s=\sum_{n=1}^{N}n\left\{\frac{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}\right\}$$

$$L_q=\sum_{n=1}^{N}(n-1)\left\{\frac{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum\limits_{n=0}^{N}\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(\frac{N-k_j}{N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}\right\}$$

(2) In case the catastrophic intensity follows the modified Binomially distribution, then we get:

$$P_n=\frac{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(1-\sum\limits_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum\limits_{n=0}^{N}\left\{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(1-\sum\limits_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}$$

$$Q_{00}=\frac{\xi}{(\lambda+\beta)}\sum_{n=1}^{N}\sum_{r=n}^{N}C_r\left(\frac{\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(1-\sum\limits_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum\limits_{n=0}^{N}\rho^n+\sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\ \prod\limits_{j=1}^{i-1}\left(\sum\limits_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod\limits_{m=0}^{i-1}\left(\sum\limits_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod\limits_{j=1}^{i}\left(\frac{\xi\left(1-\sum\limits_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}\right)$$

$$P_{00}=\frac{\left(\mu(\lambda+\beta)+\beta\xi\right)\left\{\rho+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-1)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-1)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+1}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}+\beta\xi\sum_{n=2}^{N}\sum_{r=n}^{N}C_r\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}{\lambda(\lambda+\beta)\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}$$

$$L_s=\sum_{n=1}^{N}n\left\{\frac{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}\right\}$$

$$L_q=\sum_{n=1}^{N}(n-1)\left\{\frac{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left\lceil\frac{2(N-n)-(i(i-1))}{4}\right\rceil}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}}{\sum_{n=0}^{N}\left\{\rho^n+\sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\sum_{l_0=i}^{\left\lceil\frac{2(N-n)-(i(i-1))}{4}\right\rceil}\prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right)\prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right)\rho^{L_i+n}D_i\prod_{j=1}^{i}\left(\frac{\xi\left(1-\sum_{r_j=1}^{k_j}\frac{{}^{N}C_{r_j}a^{r_j}b^{N-r_j}}{1-b^N}\right)}{\mu+\xi}\right)^{l_{(i-j)}-l_{(i-(j-1))}}\right\}}\right\}$$

7. Graphical Presentation

So far, we have obtained explicit expression for P_{00}, Q_{00}, P_n, n=1, 2… N the steady state probabilities of the system size, L_s, the mean number of customers in the system, L_q, the mean number of customers in the queue for both the cases in which the catastrophic intensity follows (I) the uniform distribution (II) the modified binomial distribution. Now here, we present some graphical presentation to highlight the effects of the catastrophe parameter ξ, the arrival rate λ and the service rate μ on P_{00}, Q_{00} and Ls for N = 8.

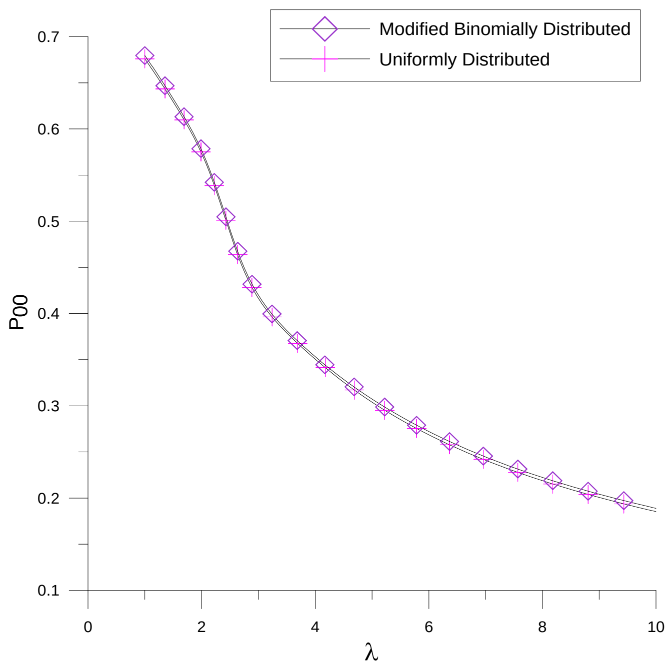

Fig.1 P_{00} as a function of λ for $\mu=5, \xi=100$

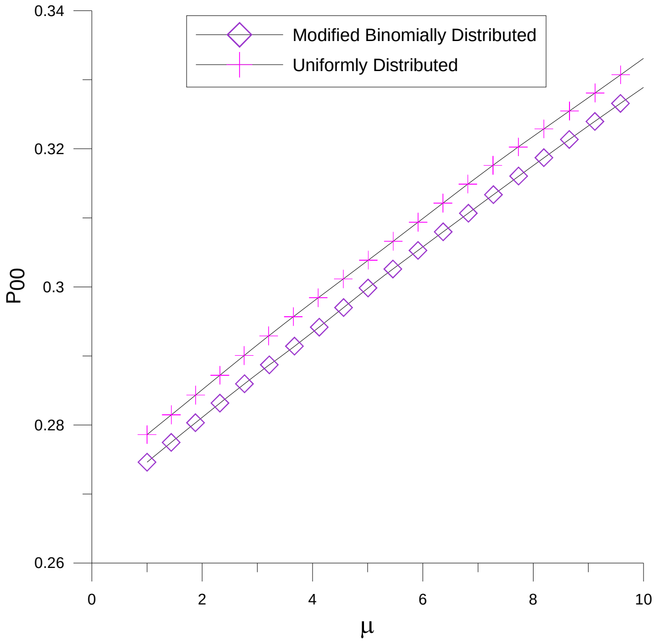

Fig.2 P_{00} as a function of μ for $\lambda = 5,\ \xi = 100$

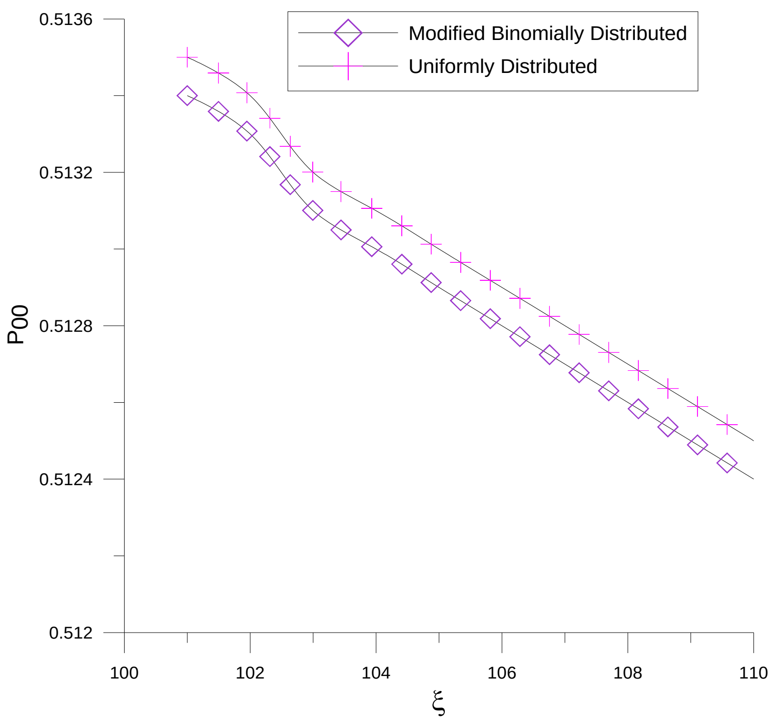

Fig.3 P_{00} as a function of ξ for $\mu = 5, \lambda = 2$

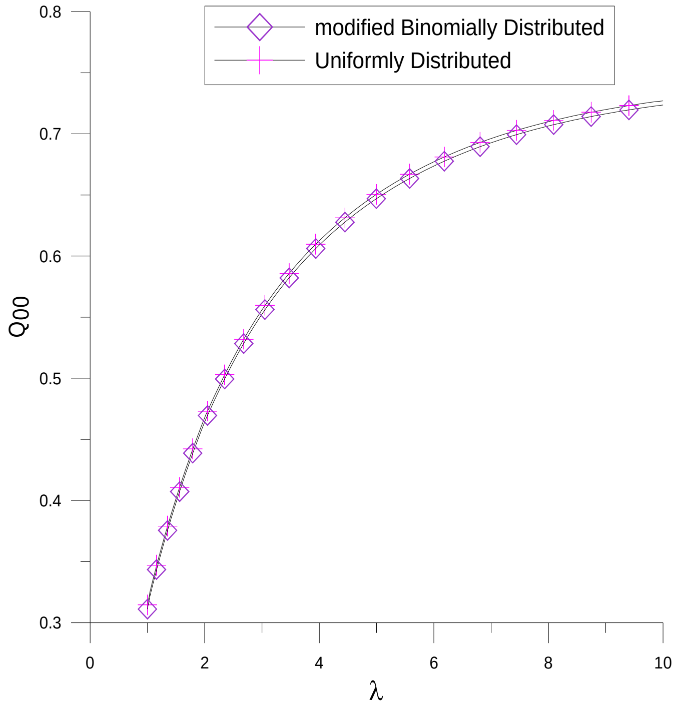

Fig.4 Q_{00} as a function of λ for $\mu = 5, \beta = 2, \xi = 100$

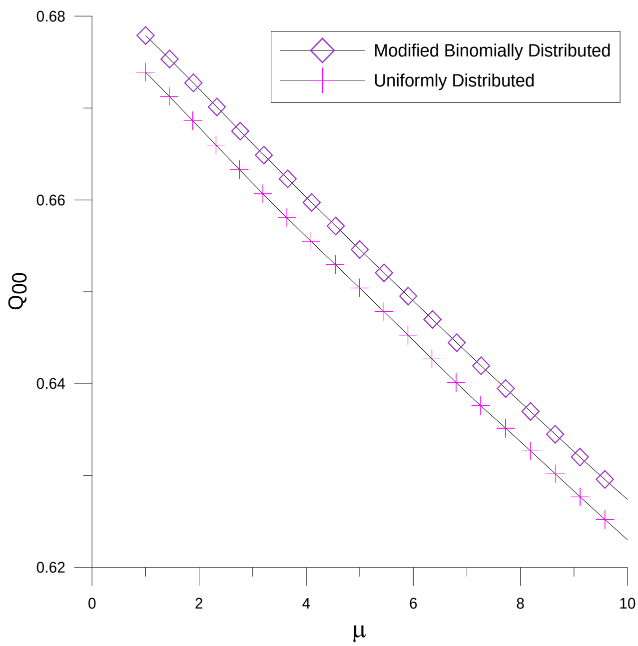

Fig.5 Q_{00} as a function of μ for $\lambda = 5, \beta = 2, \xi = 100$

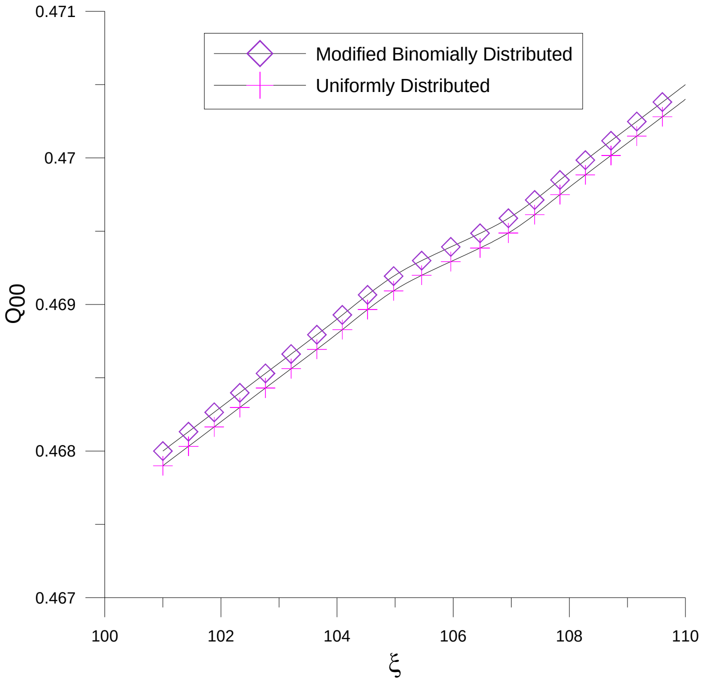

Fig.6 Q_{00} as a function of ξ for $\mu = 5, \lambda = 2, \beta = 2$

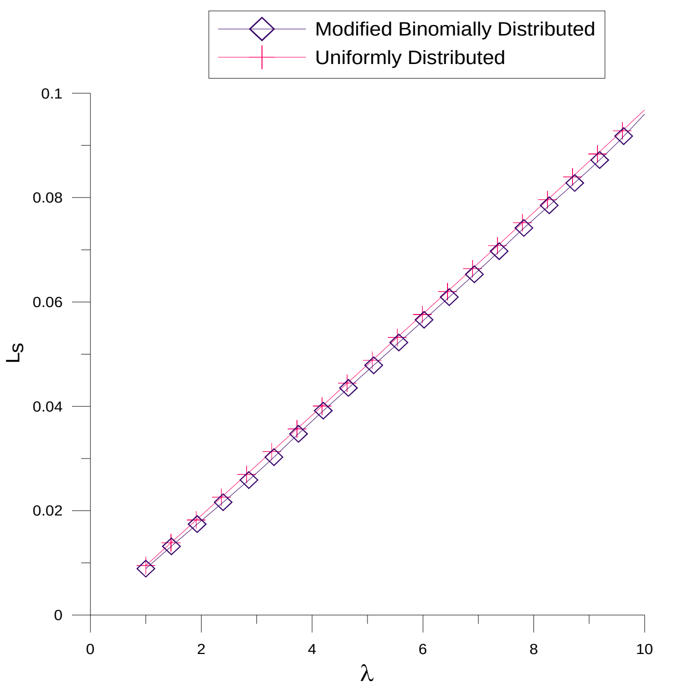

Fig.7 L_s as a function of λ for $\mu = 5, \xi = 100$

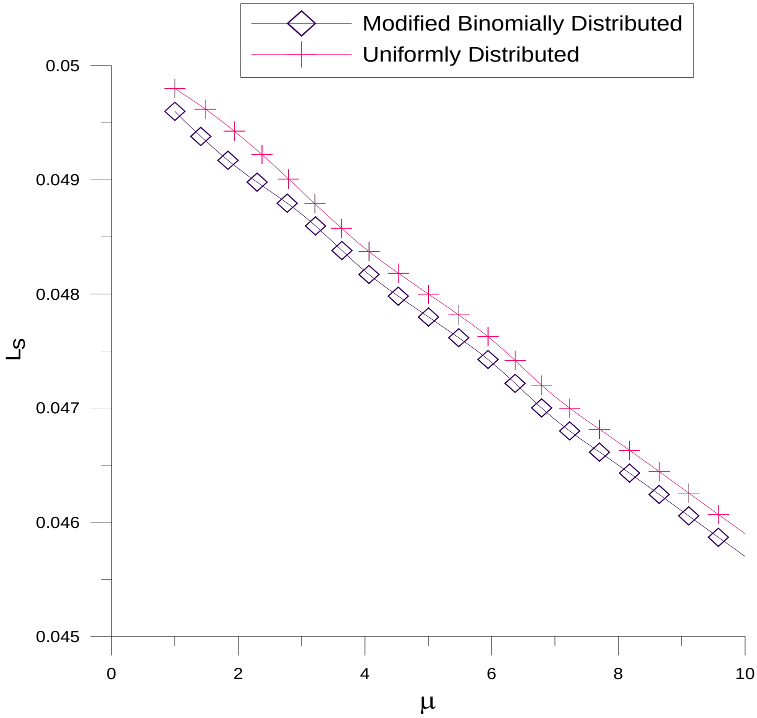

Fig.8 L_s as a function μ of for $\lambda = 5, \xi = 100$

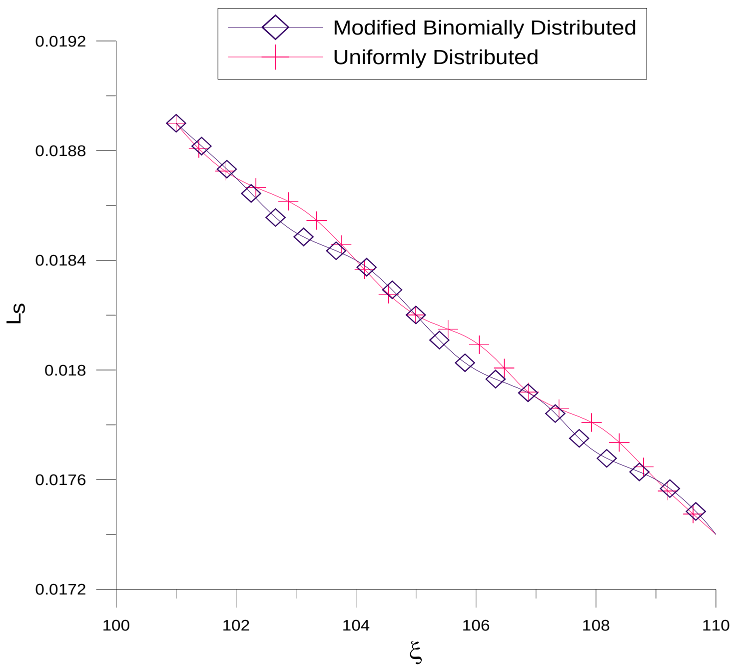

Fig.9 L_s as a function of ξ for $\mu = 5, \lambda = 2$

In figs. 1, 2 and 3, we plot the behavior of the probability $P_{00,}$ the probability of zero customers in the system without the occurrence of catastrophe, as a function of λ, μ and ξ respectively. P_{00} decreases as the arrival rate λ increases; increases with the increase in service rate μ and decreases as the catastrophe rate ξ increases. Figs. 4, 5 and 6 illustrate the effect of the arrival rate λ, service rate μ and the catastrophe rate ξ on $Q_{00,}$ the probability of zero customers in the system with the occurrence of catastrophe.. It has been observed that Q_{00} is an increasing function of λ and ξ and a decreasing function of μ. Fig. 7, 8 and 9 illustrate the effect of the arrival rate λ, service rate μ and the catastrophe rate ξ on L_s respectively. It has been noticed that L_s is an increasing function of λ and a decreasing function of μ and ξ.

8. Concluding Remarks

Queueing models after suffering some random catastrophes needs some time to regain its normal position, that time is called the restoration time. In this paper we consider that during the system failure or during the restoration time the customers may arrive and join the system. This generalization makes the system more practical. The concept of varying catastrophic intensity with restoration has numerous applications in wide variety of areas particularly in computer communication, agriculture and biosciences etc.

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