Relation between Optimal Sequences of the Bottleneck Product Rate Variation Problem with Different Objectives

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Abstract

Mixed-model just-in-time sequencing problem finds a sequence of different models distributed as evenly as possible. The sequence minimizes both the earliness and the tardiness penalties that respond to the customer demands for a variety of models without holding large inventories or incurring large shortages. The problem that deals with the sequence of the models is the product rate variation problem. The product rate variation problem minimizes the variation in the rate at which different models are produced on the assembly line.

In this paper, we consider the bottleneck case of the problem with a general objective that minimizes the maximum variation. The perfect matching with a bisection search method is used to solve the problem. A relation between optimal sequences for the problem with different objective functions is investigated.

Keywords: Product rate variation problem; sequencing problem; nonlinear integer programming

1. Introduction

Many companies use mixed-model production system to respond to the customer demands for a variety of models of a common base product. The system maintains flow line manufacturing in which a line produces a number of different models in small lots to respond to the demands. The line must incur negligible change-over costs when changing one type of model to another. Such lines are called mixed-model assembly lines in the case of assembly [6]. The change-over costs between the different models can substantially be reduced with the application of flexible workers and machinery.

Assembly lines were originated as flow oriented production systems for mass production of a single product. Nowadays, many companies have changed the assembly lines from paced single model lines for mass production to mixed-model assembly lines for mass customization of a variety of models of a common base product [1]. Mixed-model

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manufacturing gives rise to two decision problems. One, a long to mid-term planning problem i. e. the mixed-model assembly line balancing problem. The other, a short-term planning problem known as the mixed-model sequencing problem [2, 3, 4]. The mixed-model sequencing problem assigns a copy or a small batch of copies of different models to a time unit of the partitioned time horizon.

Just-in-time (JIT) production system that requires producing only the necessary products in the necessary quantities at the necessary times often uses mixed-model assembly lines [14]. The mixed-model production system under just-in-time environment is called the mixed-model just-in-time (MMJIT) production system. The most important optimization problem of the system is the problem of finding a sequence of different models distributed as evenly as possible [8, 11]. Such evenly distributed sequences are commonly known as level, balanced or fair sequences. This problem is called the mixed-model just-in-time sequencing problem (MMJITSP).

The MMJITSP minimizes both the earliness and the tardiness penalties that respond to the customer demands for a variety of models without holding large inventories or incurring large shortages. Batch sequencing procedure produces each model sequentially in a batch. With this, change-over cost can be minimized. However, inventory cost of produced models and shortage cost of models yet to be produced increase since the production is not synchronized with the demand. Level sequencing on the other hand produces either single copy or small batches of a model in order to synchronize the demand. This procedure faces high increase of change-over cost. Thus, companies reduce such cost applying flexible workers or machinery.

The MMJITSP has goals of keeping the rate of usage of parts as constant as possible and of smoothing the work overload on each workstation on the line. The first goal has been considered to be more important goal than the second one [15].

The MMJIT production system consists of a hierarchy of finite and distinct levels such as products, sub-assemblies, component parts, raw materials etc. The level in which products are manufactured is commonly known as product level or final level. The upper level is the consumption level of the outputs of the lower level and the lower level initiates a supplying process only when the upper level requires the outputs to be produced in the lower level. The sequence at the final level is crucial and affects the entire supply chain as all other levels are also inherently fixed because of the pull nature of the system. In pull system, parts from the lower level are pulled toward the upper level. The parts from raw material level are pulled forward ultimately to the product level.

The multi level nature of the production system gives rise to the multi-level MMJITSP and the phenomenon that the final level controls the other levels generates the single level MMJITSP. Minimization of the variation in the rate at which different products are produced on the line is the product rate variation problem (PRVP) [11]. This is the single-level MMJITSP. The bottleneck PRVP minimizes the maximum variation.

The bottleneck PRVP has been solved transforming it into an equivalent bottleneck assignment problem [12, 16]. The assignment problem has been solved in $O(D^3)$ time. The perfect matching with a bisection search solves the bottleneck PRVP only with absolute-deviation objective. This method is more efficient since the time it takes is O(DlogD), [17]. The problem to find the explicit bottlenecks (bounds) for the bottleneck PRVP with different objective functions remained unresolved [10]. However, an implicit bottleneck with an argument, that an optimal solution to the problem with absolute deviation is also optimal to the problem with different objectives, exists in the literature [13].

We argue that the argument is true only for feasible solution. In this paper, time window to sequence each copy of all models of any instance for the bottleneck PRVP with a general objective is investigated. The lower and the upper bottlenecks for the problem are established. The problem is solved using the perfect matching with a bisection search method. This method is more efficient than the bottleneck assignment. A relation between optimal sequences of the problem with different objective functions is investigated. The relation shows that an optimal solution to the bottleneck PRVP with an objective may not be even feasible to the problem with different objective.

The plan of the paper is as follows. Section 2 reviews the mathematical model. In Section 3, sequencing procedure is described. Section 4 establishes a relation between optimal sequences. The last section concludes the paper.

2. Mathematical Model

Given $d_i \in N$ demand for a model $i, i = 1, \dots, n$, N being the set of positive integers, with total demand $D = \sum_{i=1}^{n} d_i$ and demand ratio $r_i = \frac{d_i}{D}$, let the time horizon be partitioned into D equal units and each product is produced in a unit time. There will be k complete units of various products during the first k, $k = 1, \dots, D$ time units. Let x_{ik} be the quantity of product i produced during the time units 1 through k. Consider f_i, $i = 1, \dots, n$, unimodal symmetric convex function with minimum 0 at 0.

The mathematical model of the bottleneck PRVP [13, 15] is

(1) $\min \max f_i(x_{ik} - kr_i)$

subject to

(1.1) $\sum_{i=1}^{n} x_{ik} = k \qquad k = 1, \ldots, D$

(1.2) $x_{i(k-1)} \leq x_{ik} \qquad i = 1, \ldots, n; k = 2, \ldots, D$

(1.3) $x_{iD} = d_i;\ x_{i0} = 0 \qquad i = 1, \ldots, n$

(1.4) $x_{ik} \geq 0$, integer

Constraint (1.1) shows the cumulative production during the time units 1 through k. Constraint (1.2) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (1.3) guarantees that the demands for each product are met exactly. Constraint (1.2) and (1.4) ensure that exactly one unit of a product is scheduled during one time unit. In this paper, we consider a general objective function $F_m = |x_{ik} - kr_i|^m$, m being a positive integer.

1. Solution Procedure

The perfect matching with a bisection search appeared in [17] for the bottleneck product rate variation problem with absolute-deviation objective can also be applied for Problem F_m with necessary modifications.

The method relies on the level curves $|j - kr_i|^m$, $j = 0, 1, \ldots, d_i$; $i = 1, \ldots, n$; $k = 1, \ldots, D$ and the bottleneck (bound) $B > 0$. The level curves of all copies of the different models are drawn on the interval $[1, D]$. The interval is assumed to be a continuous time horizon though it is partitioned into D equal time units. The bottleneck is taken as a line B units above and parallel to the horizontal axis. A copy (i, j), j^{th} copy of model i, is sequenced in a time unit k in $[1, D]$ such that the level curves do not exceed B. This introduces the earliest sequencing time $E_m(i, j)$ and the latest sequencing time $L_m(i, j)$ for (i, j), $i = 1, \ldots, n$; $j = 1, \ldots, d_i$.

The sequencing times for a given B are the nearest integer of the points where the level curves intersect with not exceeding B. The earliest sequencing time satisfies the inequalities $[j - (E_m(i, j) - 1)r_i]^m > B$ and $[j - E_m(i, j)r_i]^m \leq B$ implying $E_m(i, j) = \left\lceil \frac{j - \sqrt[m]{B}}{r_i} \right\rceil$ and the latest sequencing time satisfies $[(L_m(i, j) - 1)r_i - (j - 1)]^m \leq B$ and

$[L_m(i,j)r_i - (j-1)]^m > B$ implying $L_m(i,j) = \left\lfloor \frac{j-1+\sqrt[m]{B}}{r_i} + 1 \right\rfloor$. See [9] for $m = 2$. Both $E_m(i,j)$ and $L_m(i,j)$ can be calculated in $O(D)$ time [17].

A V_1-convex bipartite graph $\mathcal{G} = (V_1 \cup V_2, \mathcal{E})$ is constructed sequencing (i,j) within $[E_m(i,j), L_m(i,j)]$, where $V_1 = \{1, \ldots, D\}$ stands for the set of sequencing times, $V_2 = \{(i,j) | i = 1, \ldots, n;\ j = 1, \ldots, d_i\}$ the set of j^{th} copy of model i and $\mathcal{E} = \{(k,(i,j)) | k \in [E_m(i,j), L_m(i,j)]\}$. A perfect matching is obtained from $\mathcal{G}$ using the earliest due date (EDD) algorithm. The algorithm sequences the lower numbered copies of a model to earlier sequencing times than the higher numbered copies [10]. Since $E_m(i,j) = \left\lceil \frac{j-\sqrt[m]{B}}{r_i} \right\rceil < \left\lceil \frac{j+1-\sqrt[m]{B}}{r_i} \right\rceil = E_m(i,j+1)$ and $L_m(i,j) = \left\lfloor \frac{j-1+\sqrt[m]{B}}{r_i} + 1 \right\rfloor < \left\lfloor \frac{j+\sqrt[m]{B}}{r_i} + 1 \right\rfloor = L_m(i,j+1)$, the perfect matching is order-preserving.

A necessary and sufficient condition for the existence of a perfect matching is $|N(K)| \geq |K|$, for all K, where $N(K) = \{(i,j) : (i,j) \in V2, \exists k \in K s.t. (k,(i,j)) \in \mathcal{E}\}$ and K is either an interval in V_1 or the neighborhood of an interval in V_1. This is Hall's theorem. The two inequalities $\sum_{i=1}^{n} \left(\lfloor k_2 r_i + \sqrt[m]{B} \rfloor - \lceil (k_1 - 1) r_i - \sqrt[m]{B} \rceil \right) \geq k_2 - k_1 + 1$ and $\sum_{i=1}^{n} \left(\lceil k_2 r_i - \sqrt[m]{B} \rceil - \lfloor (k_1 - 1) r_i + \sqrt[m]{B} \rfloor \right) \leq k_2 - k_1 + 1$ for all $k_1, k_2 \in V_1$ with $k_1 \leq k_2$ and $[E_m(i,j), L_m(i,j)] \cap [k_1, k_2] \neq \phi$ satisfy Hall's theorem for the bottleneck PRVP.

A feasible solution implies every $(i,j), i = 1, \ldots, n;\ j = 1, \ldots, d_i$ assigns a unique time unit k, $k = 1, \ldots, D$ and no time unit remains unmatched. This creates an order-preserving perfect matching in . Every order-preserving perfect matching creates a bijection $(i,j) \to k$ where $(i,j) \in V_2$ and $k \in V_1$which yields a feasible solution. Thus, an order-preserving perfect matching in G is analogous to a feasible solution.

A feasible solution with minimum B is optimal. The minimum B can be obtained using a bisection search that runs between the lower and the upper bottlenecks. Since $min(1 - r_i)^m \leq B$, the lower bottleneck is $(1 - r_{max})^m$. The upper bottleneck is $(1 - \frac{1}{D})^m$ since it satisfies the two inequalities that satisfy Hall's theorem. Minimum B can be obtained in $O(logD)$ time using a bisection search that runs in the interval $[(1 - r_{max})^m, (1 - \frac{1}{D})^m]$. Thus The time complexity to yield an optimal sequence using the bisection search is $O(DlogD)$. Every instance has optimal sequence when the given bottleneck is the upper bottleneck. However, it is not guaranteed for smaller value.

For any feasible solution, $|[kr_i] - kr_i|^m \le |x_{ik} - kr_i|^m$, $i = 1, \dots, n$, where $[kr_i]$ is the closest integer to kr_i and $|[kr_i] - kr_i|$ is $\frac{1}{2}$ for even δ_i and $\frac{\delta_i - 1}{2\delta_i}$ for odd case, where $\delta_i = \frac{D}{\gcd(\delta_i, D)}$, [7]. It is clear that the lower bottleneck for even D is $\frac{1}{2^m}1$ and less than $\frac{1}{2^m}$ for odd D. The lower bottleneck for any D is $\frac{1}{3^m}$.

No instance is feasible to Problem F_m for $B < \frac{1}{3^m}$. See [7] for m =2.

The instances of which the copies of models do not compete for the sequencing positions have the optimal value less than $\frac{1}{2^m}$. The copies of a standard instance i.e. the instance with $0 < d_1 \le \ldots \le d_n$, $gcd(d_1, \ldots, d_n) = 1, n \ge 2$ do not compete if and only if it is power-of-two. For two model case, the optimal value is less than $\frac{1}{2^m}$ if and only if the demand for one model is even and that for the other model is odd [12, 5].

2. Relation between optimal sequences

It is naturally important to establish the relationship between the optimal sequences of the bottleneck PRVP with different objective functions. Any optimal sequence if exists at the bottleneck $B = 1$ of any instance for problem Fm for some m would have been optimal for all m since all the level curves $|j - kri|^m$ meet only at $B = 1$. But the upper bottleneck $(1 - \frac{1}{D})^m$ for any problem F_m is less than 1. This shows that for any instance there may not exist the same optimal sequence for every problem F_m at the same bottleneck.

Lemma 1. The time window $T_m = [E_m(i,j), L_m(i,j)]$ follows $T_1 \subseteq T_2 \subseteq \cdots T_m$.

Proof: For $1 \le m < \acute{m}$, $E_m(i,j) = \left\lceil \frac{j - \sqrt[m]{B}}{r_i} \right\rceil = \left\lceil \frac{j - \sqrt[\acute{m}]{B}}{r_i} + \frac{\sqrt[\acute{m}]{B} - \sqrt[m]{B}}{r_i} \right\rceil \ge E_{\acute{m}}(i,j)$.
Thus, the earliest sequencing time is non-increasing function of m.
And, $L_m(i,j) = \left\lfloor \frac{j - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor = \left\lfloor \frac{j - 1 + \sqrt[\acute{m}]{B}}{r_i} + 1 + \frac{\sqrt[m]{B} - \sqrt[\acute{m}]{B}}{r_i} \right\rfloor \le L_{\acute{m}}(i,j)$.
So, the latest sequencing time is non-decreasing function of m.
We show $E_1(i,j) \le L_1(i,j)$. For $\frac{1}{2} \le B$, $E_1(i,j) = \left\lceil \frac{j - B}{r_i} \right\rceil \le \left\lfloor \frac{j - 1 + B}{r_i} + 1 + \frac{1 - 2B}{r_i} \right\rfloor \le L_1(i,j)$.
For $B < \frac{1}{2}$, (i,j) can be sequenced in the ideal position $2^{n-i}(2j-1)$ and $E_1(i,j) \le 2^{n-i}(2j-1) \le L_1(i,j)$.
Thus, $T_m \subseteq T_{\acute{m}}$, $m < \acute{m}$ holds.

The lemma shows that the time window is non-decreasing function of m.

Theorem 1. A feasible sequence to Problem F_1 is also feasible to Problem F_m, m being any positive integer.

Proof: Consider a feasible sequence s to Problem F_1. Assume that the copy (i,j) be sequenced at the time unit k. This implies $E_1(i,j) \le k \le L_1(i,j)$.
Lemma 1 shows that $E_m(i,j) \le \cdots \le E_1(i,j) \le k \le L_1(i,j) \le \cdots \le L_m(i,j)$.
Here, $E_m(i,j) \le k \Rightarrow \frac{j-\sqrt[m]{B}}{r_i} \le k \Rightarrow (j-kr_i)^m \le B$
and $k \le L_m(i,j) \Rightarrow k \le \frac{j-1+\sqrt[m]{B}}{r_i} + 1 \Rightarrow ((k-1)r_i-(j-1))^m \le B$.
Therefore, $|j-kr_i|^m \le B$. Thus, s is feasible to Problem F_m.

The same does not happen in the case of optimality. The optimality of the bottleneck PRVP differs with different objective functions.

Theorem 2. Any optimal sequence for the bottleneck PRVP with an objective function may not be optimal in the case of different objective function.

Proof : Let s be an optimal solution of any instance for Problem F_m with the bottleneck $(1-r_{max})^m$. The sequence s can not even be feasible for Problem $F_{\acute{m}}$, $\acute{m} < m$ at the same bottleneck since $(1-r_{max})^m < (1-r_{max})^{\acute{m}}$.

Theorem 2 also shows that the converse of Theorem 1is not true.

3. Conclusion

In this paper, the perfect matching with a bisection search method has been used to solve Problem F_m. It has been shown that a feasible sequence to Problem F_1 is also feasible to Problem F_m, m being positive integer. But it does not hold for optimal sequence.

Further relations between optimal sequences for the problem with different objective functions would be interesting in future research.

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